# CHAPTER 7

## RELATIVITY: THE SPECIAL THEORY

In the preceding chapters we have we have talked of space as being a set of points. This is good enough for the pure mathematician, but for geometry to be applicable to the real world, the physicist must find objects in nature that correspond to these points. A moments reflection suggest that these are events. We have in mind such idealized happenings as the collision of two particles of negligible size at a certain position in space and at a certain time. Real happenings such as the collision of two stars or two nuclei are much more complicated than this, of course, but we assume that these more complicated happenings can always be analysed into a set of these idealized events. Each event would require for its specification four numbers, the time at which the event occured and the three coordinates of the position at which it occured. This presupposes that we have available apparatus for measuring the time at which the event occured and its position relative to some chosen reference frame. We shall specify an event by the 4-tuple of numbers $x = (t, x^1, x^2, x^3) = (t, \vec{x})$ where t is the time of occurence and $\vec{x}$ is the position at which it took place.

We arrive at the notion of the universe as a 4-dimensional manifold of events, hereafter called space-time. Even at this stage some philosophical objections come to mind. Can our concepts of time and position, based as they are on macroscopic measurements with clocks and measuring rods be extrapolated to the events of elementary particle physics or cosmology? Are four dimensions sufficient for the description of the universe? We shall not discuss these questions at this point as they will recur. Ultimately, the justification for our notions must be how well theories based on them describe the universe.

## Newtonian Space-time and Gallilean Relativity

What structure shall we ascribe to space-time? First, we shall assume that it is a differentiable manifold. It seems difficult to make any progress without this assumption. This involves including as points of the manifold not only those events that actually do happen but also those that could possibly happen. Space-time, then, is a smooth manifold of both real and potential events. ~~136~~

Before 1905, when Einstein published his special theory of relativity, almost all physicists believed that space-time had the structure $E^1 \times E^3$, the Cartesian product of a 1-dimensional Euclidean time and a 3-dimensional Euclidean space. That is, $E^1$ is $R^1$ equipped with a metric such that the distance between two times $t_1$ and $t_2$ is $|t_2 - t_1|$, and $E^3$ is $R^3$ equipped with a connection and metric and with vanishing nonmetricity, torsion and curvature tensors (in short, the space we study in elementary geometry). In such a flat space it is always possible to introduce a Cartesian coordinate system and write the distance between points $\vec{x}$ and $\vec{y}$ as

$$d(\vec{x}\ ,\ \vec{y}) = [(x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2]^{1/2} \qquad (1)$$

where $x^i$ and $y^i$ are the Cartesian coordinates of the points. All events with the same value of t are said to be simultaneous. One can picture space-time as a succession of 3-dimensional Euclidean spaces, each attached to a point on the t axis as shown in Fig 1. (We have suppressed one of the coordinate axes of $E^3$ in order to draw the figure). In each of the $E^3$'s all events are simultaneous. We call this Newtonian space-time.

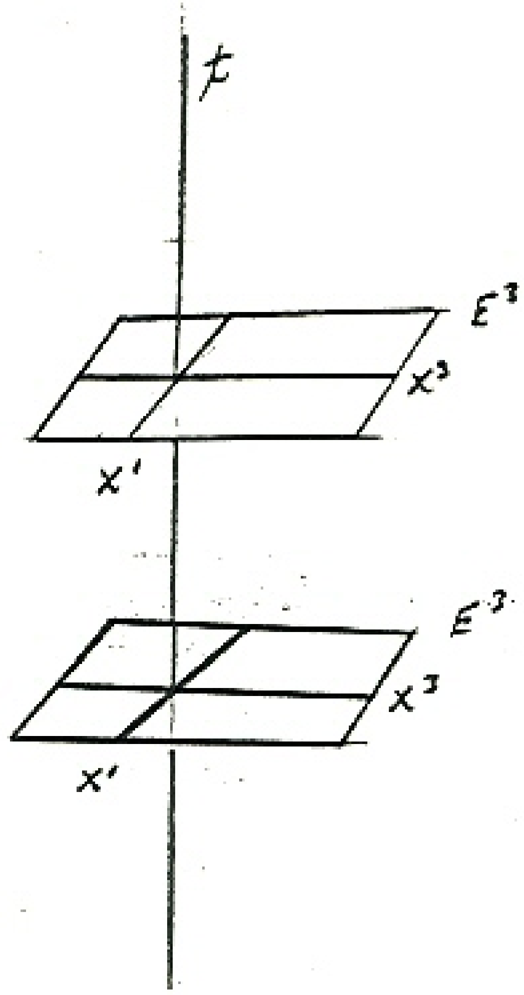


Fig.1(a).

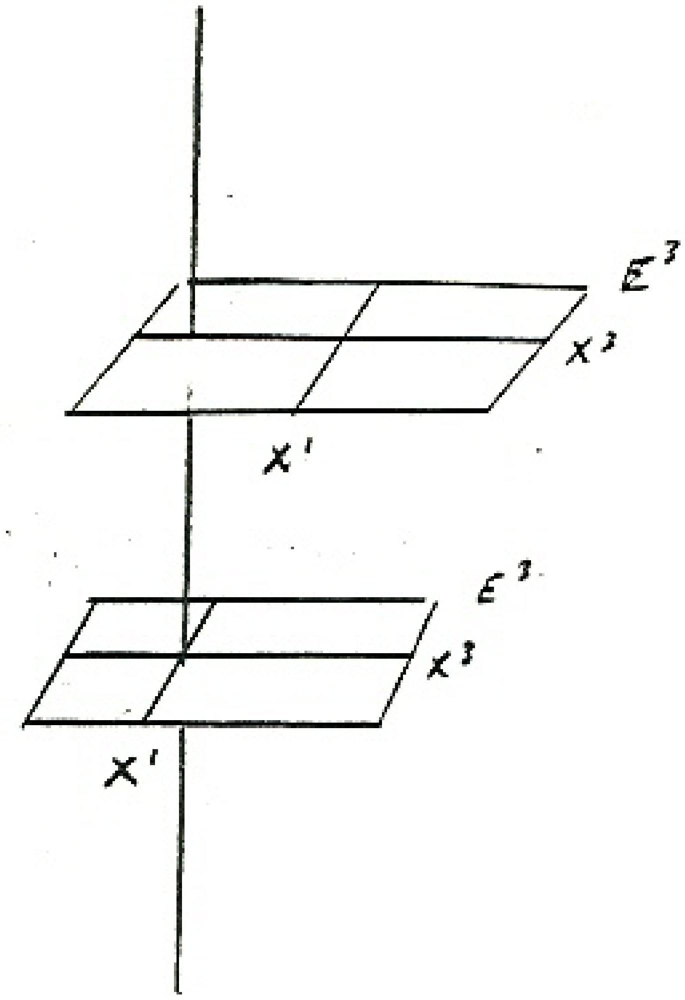


Fig.1(b).

In Fig.1(a) the point of attachment is taken to be the origin of a Cartesian coordinate system. In Fig. 1(b) the points of attachment of successive $E^3$'s are taken to be a succession of points of the $x^3$ axis. This correponds to a coordinate system whose origin is moving in the $x^3$ direction relative to the coordinate system of Fig.1(a). This is shown in Fig. 2 for a coordinate system S′ moving with a constant velocity relative to a coordinate system S. The transformation connecting the two coordinate systems is clearly

$$x^{1\prime} = x^1 \qquad (2a)$$

$$x^{2\prime} = x^2 \qquad (2b)$$

$$x^{3\prime} = x^3 - vt \qquad (2c)$$

$$t' = t \qquad (2d)$$

We have included Eq.(2d) to indicate that the points of attachment of S and S′ to the t axis are the same. This is called the <u>Gallilean transformation</u>.

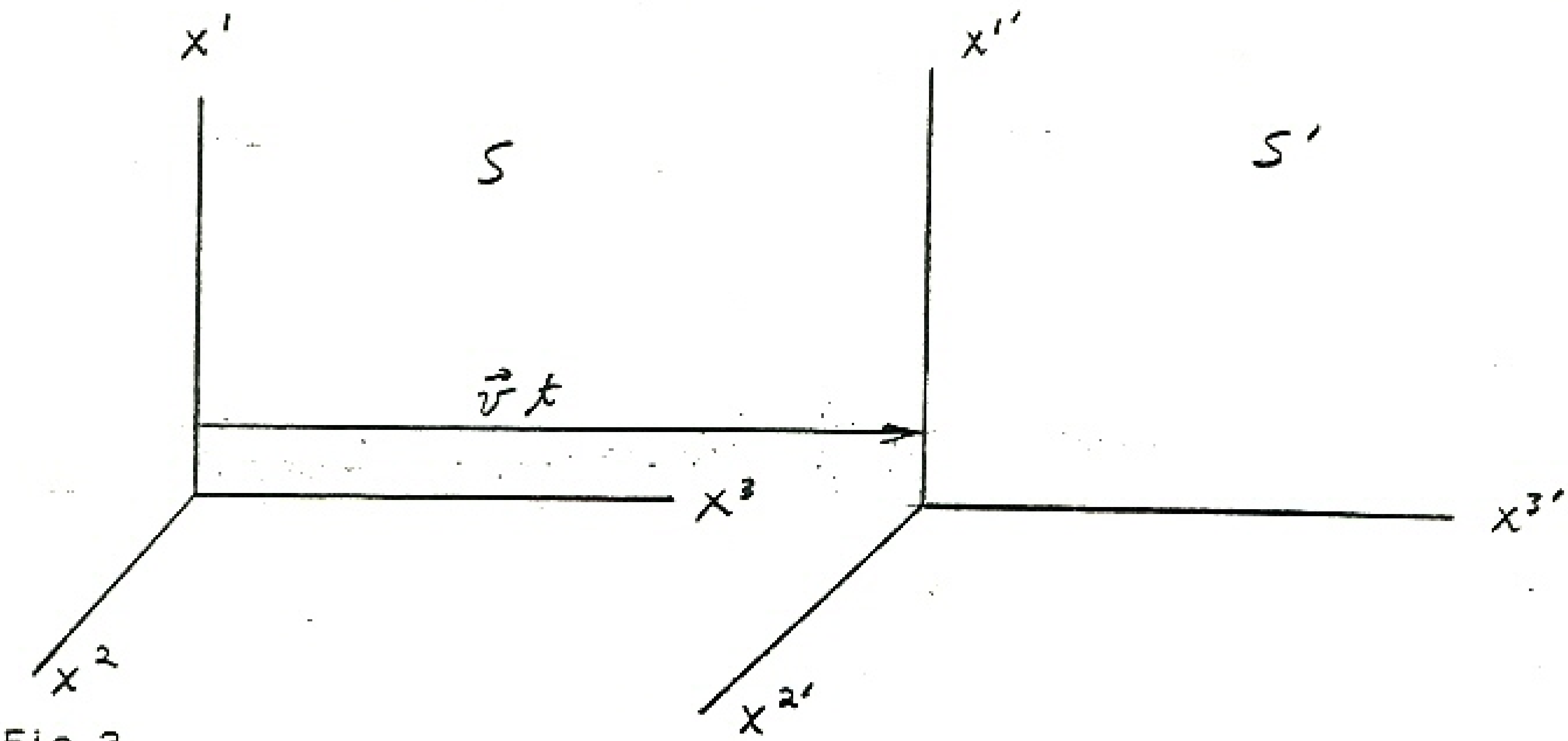


Fig.2.

More generally, we can assume that the system S′ is rotated relative to the system S and is moving with a velocity $\vec{v}$ in an arbitrary direction; also we assume that the displacement of the origin of S′ is $\vec{a}$ at the time t = 0, and that the origin on the time axis is shifted by $t_0$ . The transformation is then

$$x^{i'} = O^{i'}_{j}\, x^{j} - v^{i'} t - a^{i'} \tag{3a}$$

$$t' = t - t_0 \tag{3b}$$

or more briefly

$$\vec{x}' = O\,\vec{x} - \vec{v}t - \vec{a} \tag{4a}$$

$$t' = t - t_0 \tag{4b}$$

where O is an orthogonal matrix ($O^{-1} = O^{T}$). This is the most general form of the Gallilean transformation. These transformations form a 10 parameter group. There are three parameters necessary to specify the angle and axis of rotation, three components of $\vec{v}$, three of $\vec{a}$ and one of $t_0$. These equations may be inverted to obtain

$$\vec{x} = O^{-1}\,\vec{x} + O^{-1}\,\vec{v}(t' +- t_0) + O^{-1}\,\vec{a} \tag{5a}$$

$$t = t' + t_0 \qquad (5b)$$

Now, let us consider the equations of mechanics. We shall assume that the universe consist of a collection of particles that interact with one another through forces that depend only on the distance between the particles. The equations of motion are

$$m_\alpha\, d^2\vec{x}_\alpha/dt^2 = \sum_\beta \vec{F}_{\alpha\beta}(|\vec{x}_\alpha - \vec{x}_\beta|) \qquad (6a)$$

These equations are covariant with respect to Gallilean transformations. That is, when referred to the coordinate system S′ they become

$$m_\alpha\, d^2\vec{x}'_\alpha/dt'^2 = \sum_\beta \vec{F}'_{\alpha\beta}(|\vec{x}'_\alpha - \vec{x}'_\beta|) \qquad (6b)$$

The distance between particles is clearly invariant under Gallilean transformations, so the arguments of the forces are the same in both S and S′. When second derivatives with respect to t′ are taken, the terms in $\vec{v}t$ and $\vec{a}$ vanish. Finally, the vectors $\vec{x}$ and $\vec{F}$ transform the same way under rotations. Covariance of the equations of mechanics under displacements of the origin by $\vec{a}$ and rotations by O imply that space is homogeneous and isotropic. There is no preferred location in space, nor is there a preferred orientation. Covariance under displacement of the origin of the time axis by $t_0$ implies that there is no preferred time. Covariance under transformation to a uniformly moving coordinate system implies that all such uniformly moving systems, called <u>inertial systems</u>, are equivalent. One may imagine a number of rocket ships, each equipped with a physics laboratory and an experimental physicist, moving with constant velocities relative to one another at various locations in space and with various orientations and with unsynchronized clocks. Each of the physicists would verify that the laws of mechanics were valid in his laboratory. None could assert that there was anything special about his location in space or time or his orientation or his velocity. Democracy prevails among inertial systems.

This is not the case for systems that are accelerated with respect to inertial systems. If the term $\vec{v}t$ in Eq.(4) is replaced by $\vec{s}(t)$, then Eq.(6b) will contain the additional term $m_\alpha d^2\vec{s}/dt'^2$ not present in the equations of motion referred to the coordinate system S. This term only vanishes if $\vec{s}$ is constant or linear in t. This term acts like an additional force; it is called an inertial force. An observer in an accelerated system could detect his acceleration by the presence of inertial forces. It is by the presence or absence of inertial forces that noninertial and inertial coordinate systems may be distinguished. Note that inertial forces are proportional to the mass of the particle acted upon. The same is true of gravitational forces, and this common feature will play an important role in the general theory of relativity.

We have just seen that the laws of mechanics are covariant under Gallilean transformations. This is the <u>Gallilean principle of relativity</u>. Absolute positions in space-time and absolute velocities in space-time are meaningless concepts, as far as the laws of mechanics are concerned, since in principle they can not be observed. However, with the developement of electromagnetic theory in the latter half of the nineteenth century, the possibility appeared of detecting an absolute velocity by electromagnetic and optical experiments.

Consider Maxwell's equations for the electric and magnetic fields $\vec{E}$ and $\vec{B}$.

$$\nabla\cdot\vec{E} = 4\pi\rho \tag{7a}$$

$$\nabla\cdot\vec{B} = 0 \tag{7b}$$

$$\nabla\times\vec{E} = -(1/c)\frac{\partial\vec{B}}{\partial t} \tag{7c}$$

$$\nabla\times\vec{B} = (4\pi/c)\vec{J} + (1/c)\frac{\partial\vec{E}}{\partial t} \tag{7d}$$

As is well known these equations predict the existence of electromagnetic waves that travel with the velocity of $c = 3 \times 10^{10}$ cm./sec., the velocity of light. But, what is this velocity relative to? If the waves move with velocity $\vec{c}$ in one coordinate system, then in a coordinate system that moves with velocity $\vec{v}$ relative to the first, the waves must have a velocity $\vec{c} - \vec{v}$. Maxwell's equations cannot be covariant with respect to Gallilean transformations. They must take a different form in moving reference systems to take account of the different wave velocities in these systems. Only in one system, the stationary system, would the velocity of light be the same in all directions and Maxwell's equations take the simple form given above. This stationary system was identified with a system that was at rest with respect to the ether, a hypothetical medium through which the waves propagated. Experience with sound waves in fluids and elastic waves in solids had accustomed the physicists of the nineteenth century to think in terms of waves propagating through a medium whose properties determined the velocity of propagation.

Ingeneous optical and electromagnetic experiments were performed to measure the velocity of the earth relative to the ether. Rather surprisingly, that velocity was found to be zero within the limits of experimental accuracy. The conclusion seemed to be unescapable that the velocity of light was constant and independent of direction in all reference systems.

## THE LORENTZ TRANSFORMATION

How must the coordinates and time in a reference system S be related to those in a system S′, moving with respect to the first, if the velocity of light is to be constant and independent of direction in both systems. Let us consider the reference systems S and S′ of Fig.2 whose origins coincide at the time $t = t' = 0$ and suppose at this time a flash of light is emitted from the origin. Observers in S and S′ would see spherical waves expanding from the origins of their coordinate systems with velocity c. At the time t the observer in S would write the equation for the wave front as

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{8a}$$

and the observer in S′ would write the equation for the same wave front as

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{8b}$$

Clearly this cannot be satisfied for $t' = t$, for $x' \neq x$.
We wish to find relations between $(x, y, z, t)$ and $(x', y', z', t')$ such that Eq.(8a) implies (8b).

We can simplify yhe notation by defining $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ and similar expressions for the primed variables, and then we write Eqs.(8a,b) as

$$\eta_{\alpha\beta}x^\alpha x^\beta = 0 = \eta_{\alpha'\beta'}x^{\alpha'}x^{\beta'} \tag{9}$$

where Greek indices will henceforth be understood to take on the values 0,1,2,3, and $\eta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1$ Eq.(9) will be satisfied if

$$\eta_{\alpha'\beta'}x^{\alpha'}x^{\beta'} = K(\vec{v})\eta_{\alpha\beta}x^\alpha x^\beta \tag{10}$$

where K may depend on the relative velocity of the two reference systems. Because of the isotropy of space we expect K to depend only on the magnitude of the velocity, v. We could equally well regard the transformation as being from S′ to S which moves with the velocity $-\vec{v}$ relative to S′; then we would write

$$\eta_{\alpha\beta}x^\alpha x^\beta = K(-\vec{v})\eta_{\alpha'\beta'}x^{\alpha'}x^{b'} \tag{11}$$

Combining Eqs.(10) and (11) we obtain

$$K(-\vec{v})K(\vec{v}) = K^2(v) = 1 \tag{12a}$$

$$K = \pm 1 \tag{12b}$$

Since when $\vec{v} = 0$ the two reference systems coincide, $K(0) = 1$ and the positive sign in Eq.(12b) must be chosen.

We shall assume a linear relationship between the primed and unprimed space-time coordinates; thus

$$x^{\mu'} = L^{\mu'}_{\ \nu} x^{\nu} \tag{13a}$$

$$x^{\nu} = L^{\nu}_{\mu'} x^{\mu'} \tag{13b}$$

$$L^{\mu'}_{\ \alpha} L^{\alpha}_{\nu'} = \delta^{\mu'}_{\ \nu'} \tag{13c}$$

Substituting into Eq.(10) with K =1, gives

$$\eta_{\mu'\nu'} = L_{\mu'}^{\ \alpha} L_{\nu'}^{\ \beta} \eta_{\alpha\beta} \tag{14}$$

These are the relations that the elements of the transformation matrix L must satisfy in order that the velocity of light be constant and independent of direction in all inertial systems. Transformations that satisfy Eq.(14) are called <u>Lorentz transformations</u>.

We shall find the transformation matrix for the case of relative motion parallel to the z-axis as in Fig.2. Assume that the coordinates perpendicular to $\vec{v}$ are unchanged, so

$$x^{1'} = x^{1} \tag{15a}$$

$$x^{2'} = x^{2} \tag{15b}$$

$$x^{3'} = L^{3'}_{\ 0} x^{0} + L^{3'}_{\ 3} x^{3} \tag{15c}$$

$$x^{0'} = L^{0'}_{\ 0} x^{0} + L^{0'}_{\ 3} x^{3} \tag{15d}$$

Using

$$\eta_{\mu\nu} = L^{\alpha'}_{\mu} L^{\beta'}_{\ \nu} \eta_{\alpha'\beta'} \tag{16a}$$

for $(\mu,\nu) = (0,0)$, $(3,3)$ and $(0,3)$ gives

$$1 = (L_0^{\ 0'})^2 - (L_0^{3'})^2 \tag{16b}$$

$$-1 = (L_3^{\ 0'})^2 - (L_3^{3'})^2 \tag{16c}$$

$$0 = L_0^{0'} L_3^{0'} - L_0^{3'} L_3^{3'} \tag{16d}$$

Now consider the origin of S'. It must move with the velocity v in the system S, so

$$x^{3'} = 0 = L_0^{3'} x^0 + L_3^{3'} x^3$$

$$= L_0^{3'} ct + L_3^{3'} vt \tag{16e}$$

After a bit of algebra we find

$$L_\beta^{\alpha'} = \begin{bmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{bmatrix} \tag{17a}$$

where

$$\cosh\phi = (1 - v^2/c^2)^{-1/2} \tag{17b}$$

$$\sinh\phi = (v/c)(1 - v^2/c^2)^{-1/2} \tag{17c}$$

We shall write out the transformation from (x,y,z,t) to (x',y',z',t') for easy reference

$$x' = x$$

$$y' = y$$

(18)

$$z' = \frac{z - vt}{(1 - v^2/c^2)^{1/2}}$$

$$t' = \frac{t - (v/c^2)z}{(1 - v^2/c^2)^{1/2}}$$

This result is easily generalized to a general direction of the velocity $\vec{v}$. We note that the components of $\vec{x}$ perpendicular to $\vec{v}$ are unchanged, so we divide $\vec{x}$ into its parallel and transverse parts; thus

$$\vec{x} = \vec{x}_{||} + \vec{x}_{\perp} \tag{19a}$$

where

$$\vec{x}_{||} = \vec{m}\vec{m}.\vec{x} \tag{19b}$$

$$\vec{x}_{\perp} = (1 - \vec{m}\vec{m})\cdot\vec{x} \tag{19c}$$

and $\vec{m} = \vec{v}/v$ is a unit vector, and 1 denotes the unit tensor. In a similar way $\vec{x}'$ is divided into its parallel and transverse parts. Then

$$\vec{x}'_{||} = \vec{x}_{||} \cosh\phi - \vec{m} \sinh\phi \; x^0 \tag{20a}$$

$$\vec{x}_{\perp}' = \vec{x}_{\perp} \tag{20b}$$

and the general transformation formulas are

$$\vec{x}' = [1 - \vec{m}\vec{m}(1 - \cosh\phi)]\cdot\vec{x} - \vec{m} \sinh\phi \; x^0 \tag{21a}$$

$$x^{0\prime} = x^0 \cosh\phi - \vec{m}\cdot\vec{x} \sinh\phi \tag{21b}$$

In a more general form of the Lorentz transformation the coordinate axes are rotated as well as having the velocity of translation changed. The form of the Lorentz transformation discussed above in which there is a change on velocity but no rotation is known as a boost. The transformation corresponding to a pure rotation through an angle $\theta$ about an axis $\vec{n}$ was given in Chapter 2 as

$$\vec{x}' = [1 \cos\theta + \vec{n}\vec{n}(1 - \cos\theta)]\cdot\vec{x} + (\vec{n} \times \vec{x}) \sin\theta \qquad (22a)$$

$$x^{0'} = x^0 \qquad (22b)$$

A more general Lorentz transformation can be written as the product of a boost and a rotation. From Eqs.(21) and (22) one can identify the elements of the transformation matrix. For boosts these are:

$$L^{0'}_{0} = \cosh\phi \qquad (23a)$$

$$L^{0'}_{i} = - m_i \sinh\phi \qquad (23b)$$

$$L^{i'}_{0} = - m^{i'} \sinh\phi \qquad (23c)$$

$$L^{i'}_{j} = \delta^{i'}_{j} + m^{i'} m_j (\cosh\phi - 1) \qquad (23d)$$

For rotations the elements of the transformation matrix are:

$$L^{0'}_{0} = 1 \qquad (24a)$$

$$L^{0'}_{i} = L^{i'}_{0} = 0 \qquad (24b)$$

$$L^{i'}_{j} = \delta^{i'}_{j} \cos\theta + n^{i'} n_j (1 - \cos\theta) + \sin\theta\, \epsilon^{i'}{}_{jk} n^k \qquad (24c)$$

Now, we return to the general form of the Lorentz equation and write Eq.(13a) as the matrix equation

$$x' = L\, x \qquad (25)$$

with the understanding that x is a column vector with elements

$x^0$, $x^1$, $x^2$, $x^3$ and similarly for $x'$, and L is a square matrix. We may write

$$\eta_{\mu\nu} = L^{\alpha'}{}_{\mu} L^{\beta'}{}_{\nu} \eta_{\alpha'\beta'} \tag{26a}$$

as

$$\eta = L^T \eta L \tag{26b}$$

Taking the determinant of this equation and using the rule that the determinant of a matrix product is the product of the determinants gives

$$\det(\eta) = \det(L^T)\det(\eta)\det(L) \tag{27a}$$

from which

$$[\det(L)]^2 = 1 \tag{27b}$$

$$\det(L) = \pm 1 \tag{27c}$$

This divides the Lorentz transformations into two classes, the <u>proper</u> transformations for which $\det(L) = 1$ and the <u>improper</u> transformations for which $\det(L) = -1$.

Letting $(\mu,\nu) = (0,0)$ in Eq.(26a), we obtain

$$1 = L^{\alpha'}{}_{0} L^{\beta'}{}_{0} \eta_{\alpha'\beta'}$$

$$= (L^{0'}{}_{0})^2 - (L^{1'}{}_{0})^2 - (L^{2'}{}_{0})^2 - (L^{3'}{}_{0})^2 \tag{28a}$$

from which

$$(L^{0'}{}_{0})^2 \geq 1 \tag{28b}$$

This divides the Lorentz transformations into two classes; those for which $L^{0'}_{\ 0} \geq 1$, called <u>orthochronous</u>, and those for which $L^{0'}_{\ 0} \leq -1$, called <u>antichronous</u>. Altogether, the Lorentz transformations are divided into four classes which are labeled as follows:

$$L_+^\uparrow \quad \det(L) = +1, \quad L^{0'}_{\ 0} \geq 1, \text{ proper, orthochronous} \qquad (29a)$$
$$L_-^\uparrow \quad \det(L) = -1, \quad L^{0'}_{\ 0} \geq 1, \text{ improper, orthochronous} \qquad (29b)$$
$$L_+^\downarrow \quad \det(L) = +1, \quad L^{0'}_{\ 0} \leq -1, \text{ proper, antichronous} \qquad (29c)$$
$$L_-^\downarrow \quad \det(L) = -1, \quad L^{0'}_{\ 0} \leq -1, \text{ improper, antichronous} \qquad (29d)$$

It is easily shown that the Lorentz transformations form a group. Only the part $L_+^\uparrow$ is a subgroup since it is the only part that contains the identity. The rotations and boosts are subgroups of $L_+^\uparrow$.

If we replace the angle $\phi$ in Eqs.(21) by an infinitesimal angle $\delta\phi$ and replace the angle $\theta$ in Eqs.(22) by the infinitesimal angle $\delta\theta$, then we get the transformations for an infinitesimal boost and an infinitesimal rotation given below.

$$x' = (1 - \delta\phi\vec{m}\cdot\vec{M})\ x \qquad (30a)$$

$$x' = (1 + i\delta\theta\vec{n}\cdot\vec{N})\ x \qquad (30b)$$

where the vectors $\vec{M}$ and $\vec{N}$ have components that are the following matrices:

$$N^1 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}$$

$$N^2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{vmatrix}$$

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{matrix}$$

$$N^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$M^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \qquad (30c)$$

$$M^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$M^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

We can combine the infinitesimal boosts and rotations to get a general infinitesimal transformation belonging to $L_+^\uparrow$; thus

$$x' = L\,x \qquad (31a)$$

where

$$L = 1 - \delta\phi\,\vec{m}\cdot\vec{M} + i\delta\theta\,\vec{n}\cdot\vec{N} \qquad (31b)$$

We can obtain finite transformations corresponding to angles $\phi$ and $\theta$ by the following process. We divide $\phi$ and $\theta$ by the very large number N and let $\delta\phi = \phi/N$ and $\delta\theta = \theta/N$. Then we apply the infinitesimal transformation N times to get the an approximation to the finite transformation. In the limit that

N becomes infinite this should be exact, so

$$L(\vec{m},\phi;\vec{n},\theta) = \lim_{N\to\infty} [1 + (1/N)(-\phi\vec{m}\cdot\vec{M} + i\theta\vec{n}\cdot\vec{N})]^{N}$$

$$= \exp(-\phi\vec{m}\cdot\vec{M} + i\theta\vec{n}\cdot\vec{N}) \qquad (32)$$

This is the Lorentz transformation matrix for a transformation consisting of a rotation through an angle $\theta$ about an axis $\vec{n}$ and transformation to a set of axes moving in the direction $\vec{m}$ with velocity given in terms of the angle $\phi$ by Eq.(17).

The lorentz transformations discussed above are more properly called <u>homogenous Lorentz transformations</u>. They constitute a six parameter group, the <u>homogenous Lorentz group</u> with parameters $\vec{m}$, $\phi$, $\vec{n}$, $\theta$. The group may be enlarged by adding space time displacements a with components $a^{\mu}$. These transformations are

$$x^{\mu'} = L^{\mu'}{}_{\alpha}\, x^{\alpha} + a^{\mu'} \qquad (33a)$$

or more briefly

$$x' = L\, x + a \qquad (33b)$$

This ten parameter group is called the <u>inhomogeneous Lorentz group</u> or the <u>Poincare' group</u>.

<u>Problem 1.</u>

Use the power series for the exponential and show that

$$\exp(-\phi\vec{m}\cdot\vec{M}) = 1 + (\vec{m}\cdot\vec{M})^2 (\cosh\phi - 1) - \vec{m}\cdot\vec{M} \sinh\phi$$

$\exp(i\theta\vec{n}\cdot\vec{N}) =$ ~~$1 \cos\theta + \hat{n}\hat{n}(1 - \cos\theta) + i\vec{n}\cdot\vec{N} \sin\theta$~~

$= 1 + (\vec{n}\cdot\vec{N})^2 (\cos\theta - 1) - \vec{n}\cdot\vec{N} \sin\theta$ ERROR

and that these are the same as Eqs.(21) and (22).

MINKOWSKI SPACE-TIME

Consider two events x and y with space time coordinates $x^\mu = (x^0 = ct_x, \vec{x})$ and $y^\mu = (y^0 = ct_y, \vec{y})$. The quantity

$$s(x, y)^2 = (x^0 - y^0)^2 - |\vec{x} - \vec{y}|^2$$

$$= \eta_{\alpha\beta}(x^\alpha - y^\alpha)(x^\beta - y^\beta) \qquad (34a)$$

is invariant under Lorentz transformations. For an infinitesimal separation with $y^\mu = x^\mu + dx^\mu$ it is

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta \qquad (34b)$$

We shall interpret s(x, y) as the space-time separation of the two events. If the two events occur at the same point in space ($\vec{x} = \vec{y}$) then s is their separation in time. If they occur at the same time ($x^0 = y^0$), then their spatial separation is $|s|$. We interpret Eq.(34b) as the metric for the space and $\eta_{\alpha\beta}$ as the metric tensor. For lack of a compelling reason to assume otherwise, we assume the torsion and nonmetricity tensors vanish. Then, The curvature tensor also vanishes since there are coordinate systems in which the components of the metric tensor have the constant values $\eta_{\alpha\beta}$. In these coordinate systems the components of the connection vanish and the equations of geodesics are

$$d^2x^\mu/ds^2 = 0 \qquad (35a)$$

with solutions

$$x^\mu = a^\mu s + b^\mu \qquad (35b)$$

where $a^\mu$ and $b^\mu$ are constants. These are straight lines. If s is eliminated between the equations for $x^0$ and $\vec{x}$, one obtains

$$\vec{x} = \vec{a}(ct - b^0)/a^0 + \vec{b} \qquad (36)$$

This is the path followed by a free particle, according to Newton's first law of motion, so we shall interpret space-time geodesics as the trajectories of free particles. We have included among these trajectories those of particles that move with a velocity greater than c; this will be reexamined when particle mechanics is discussed. This space-time with vanishing torsion, nonmetricity and curvature tensors and signature (1,3) (that is, when the metric tensor is reduced to diagonal form, one diagonal element is positive and three are negative) is called Minkowski space-time and is denoted by $M_4$. It is said to be a semi-Riemannian space, since a Riemannian space has a positive definite metric. Minkowski space-time is the space-time of the special theory of relativity.

Consider a space-time point p. Choose base vectors $\bar{e}_\mu$ at p with the scalar product $g(\bar{e}_\mu, \bar{e}_\nu) = \eta_{\alpha\beta}$. These vectors span the tangent space at p. We may choose p as the origin of our coordinate system, and then the position of any event is given by a vector $\bar{x} = x^\mu \bar{e}_\mu$.

A vector $\bar{A}$ in the tangent space at p has the "length$^2$"

$$|\bar{A}|^2 = \eta_{\alpha\beta} A^\alpha A^\beta \qquad (37)$$

This may be positive, negative or zero. Therefore the vectors $\bar{A}$ in the tangent space at a point may be divided into three classes; those with $|\bar{A}|^2 > 0$ called time-like vectors, those with $|\bar{A}|^2 < 0$ called space-like vectors and those with $|\bar{A}|^2 = 0$ called light-like vectors. Time-like vectors with $A^0 > 0$ are said to be directed toward the future, while those with $A^0 < 0$ are said to be directed toward the past. The regions of the tangent space in which these vectors lie are shown in Fig. 3. We have suppressed one spatial dimension in order to draw the picture.

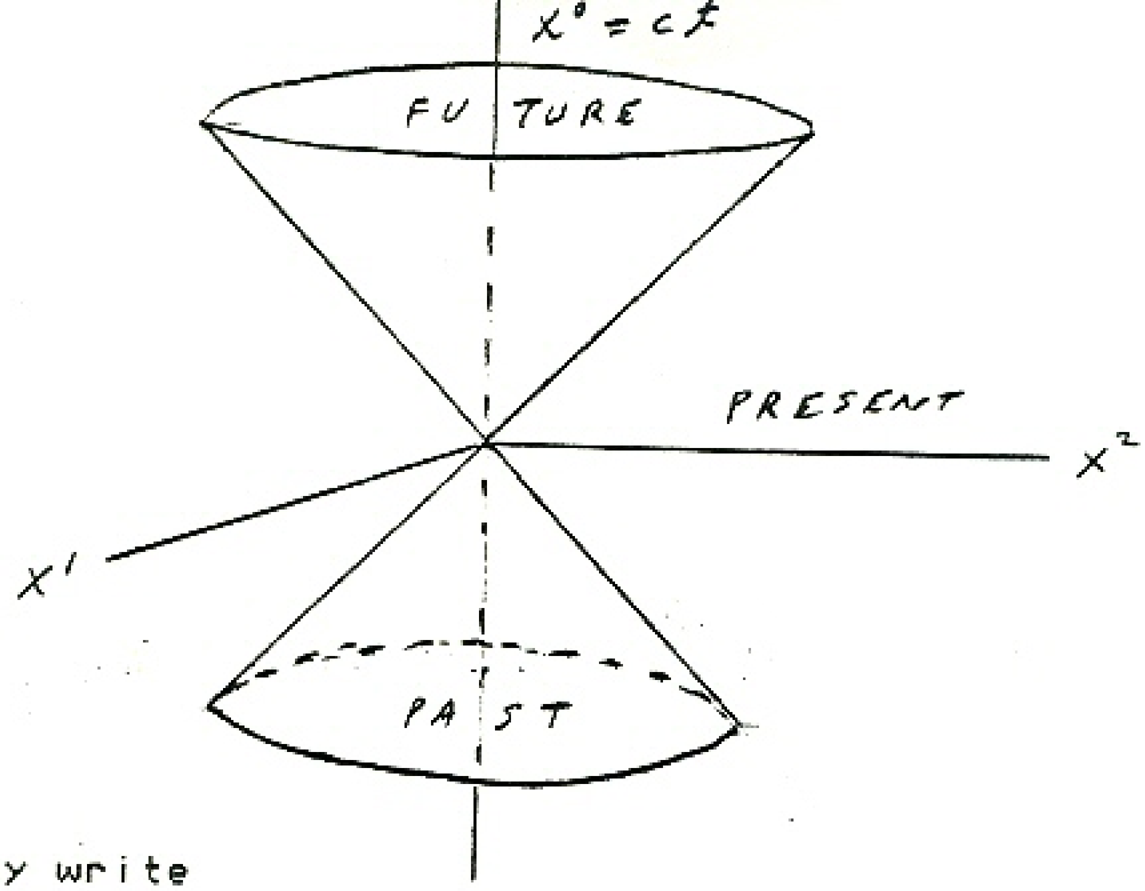


Fig. 3.

We may write

$$|\bar{A}|^2 = (A^0)^2 - |\vec{A}|^2 \begin{cases} > 0, \text{ time-like} \\ = 0, \text{ light-like} \\ < 0, \text{ space-like} \end{cases} \quad (38)$$

Light-like vectors lie on the surface of the double cone $A^0 = \pm|\vec{A}|$. This is called the light cone. Time-like vectors lie in the interior of the light cone, with future directed vectors within the positive light cone and past directed vectors within the negative light cone. Space-like vectors lie outside of the light cone.

Let us imagine an observer S carrying with him an inertial system whose origin passes through the space-time point p at the time t = 0. The observer can recognize that he is in an inertial system by the absence of inertial forces. For this observer, as he passes through p, future events are those with position vectors $\bar{x}$ in the future light cone and past events are those with position vectors in the past light cone. The tangent vector to his space-time path is $\bar{e}_0$. Events whose position vectors are orthogonal to $\bar{e}_0$ are simultaneous with his passage through p. Since he passes through p at the time $x^0$, these simultaneous events have coordinates $(0, \vec{x})$. Let us imagine a second observer S′, moving with velocity $\vec{v}$ relative to S and carrying with him his inertial system whose origin passes through the point p at time t′ = 0. We shall assume that the spatial coordinate systems of S and S′ are not rotated relative to one another, so that the two space-time coordinate systems are related to one another by Eqs.(21). The events that are simultaneous with the passage of the origin through p for observer S′ are those with $x^{0'} = 0$. However, these events are not simultaneous for observer S, for setting $x^{0'} = 0$ in Eq.(21b) gives

$$0 = x^0 - \vec{v}\cdot\vec{x}/c \tag{39}$$

which is not satisfied for $x^0 = 0$ unless $\vec{x}\cdot\vec{v} = 0$. We see that observers in relative motion would disagree about the simultaneity of events.

Observers in relative motion would disagree about lengths and time intervals. Let us imagine a measuring rod alligned parallel to the z′ axis of the coordinate system of observer S′. This observer would measure its length to be

$$z'_{(2)} - z'_{(1)} = L_0 \tag{40a}$$

where $z'_{(1)}$ and $z'_{(2)}$ are the coordinates of the ends of the rod. From Eq.(18) we find

$$L_0 = [(z_{(2)} - z_{(1)}) - v(t_{(2)} - t_{(1)})]/(1 - v^2/c^2)^{1/2} \tag{40b}$$

To obtain a meaningful result for his measurement of the length of the rod, observer S should measure the positions of the ends of the rod simultaneously. Setting $t_{(1)} = t_{(2)}$, we find L, the length of the rod as measured by S to be

$$L = z_{(2)} - z_{(1)} = L_0(1 - v^2/c^2)^{1/2} \tag{40c}$$

The observer S measures a length L that is shorter than the length $L_0$ measured by S′.

Next, consider two events that occur at the same position in the S′-system. These may be two ticks of a clock that is stationary in the coordinate system of observer S′, for instance. The observer S′ will measure a time interval $T_0$ between them; the observer S will measure a time interval T. To calculate the relation between $T_0$ and T we need the inverse of Eq.(18). This may be done by solving for $\vec{x}$ and t in terms of $\vec{x}'$ and t', but the easiest way is just to reverse the sign of v obtaining

$$x = x'$$

$$y = y'$$

$$z = \frac{z' + vt'}{(1 - v^2/c^2)^{1/2}} \tag{41}$$

$$t = \frac{t' + (v/c^2)z'}{(1 - v^2/c^2)^{1/2}}$$

From the last of these we obtain

$$T = t_{(2)} - t_{(1)}$$

$$= [(t'_{(2)} - t'_{(1)}) + (v/c^2)(z'_{(2)} - z'_{(1)})]/(1 - v^2/c^2)^{1/2} \tag{42a}$$

Since the two events occur at the same place in the S'-system, $z'_{(1)} = z'_{(2)}$; therefore

$$T = T_0/(1 - v^2/c^2)^{1/2} \tag{42b}$$

Observer S measures a longer time interval than doer observer S'.

It is meaningful to speak of lengths and time intervals only with the understanding that these are measured in a referense system in which the measured object is at rest. We call these proper lengths and proper times and have distinguished them by the subscript zero on $L_0$ and $T_0$. If we speak of the radius or lifetime of an unstable nucleus, we must understand that these are to be measured with the nucleus at rest. Otherwise we obtain a variety of different results that depend on the velocity of the observer.

The trajectory of a particle is a sequence of events that lie along some path in space-time. We call this the particle's world line. This is a curve in 4-dimensional space-time. In order to sketch it on a 2-dimensional sheet of paper we may plot three graphs of $x^i$ versus t (or $x^0$) for i = 1,2,3, as we have done in Fig.4.

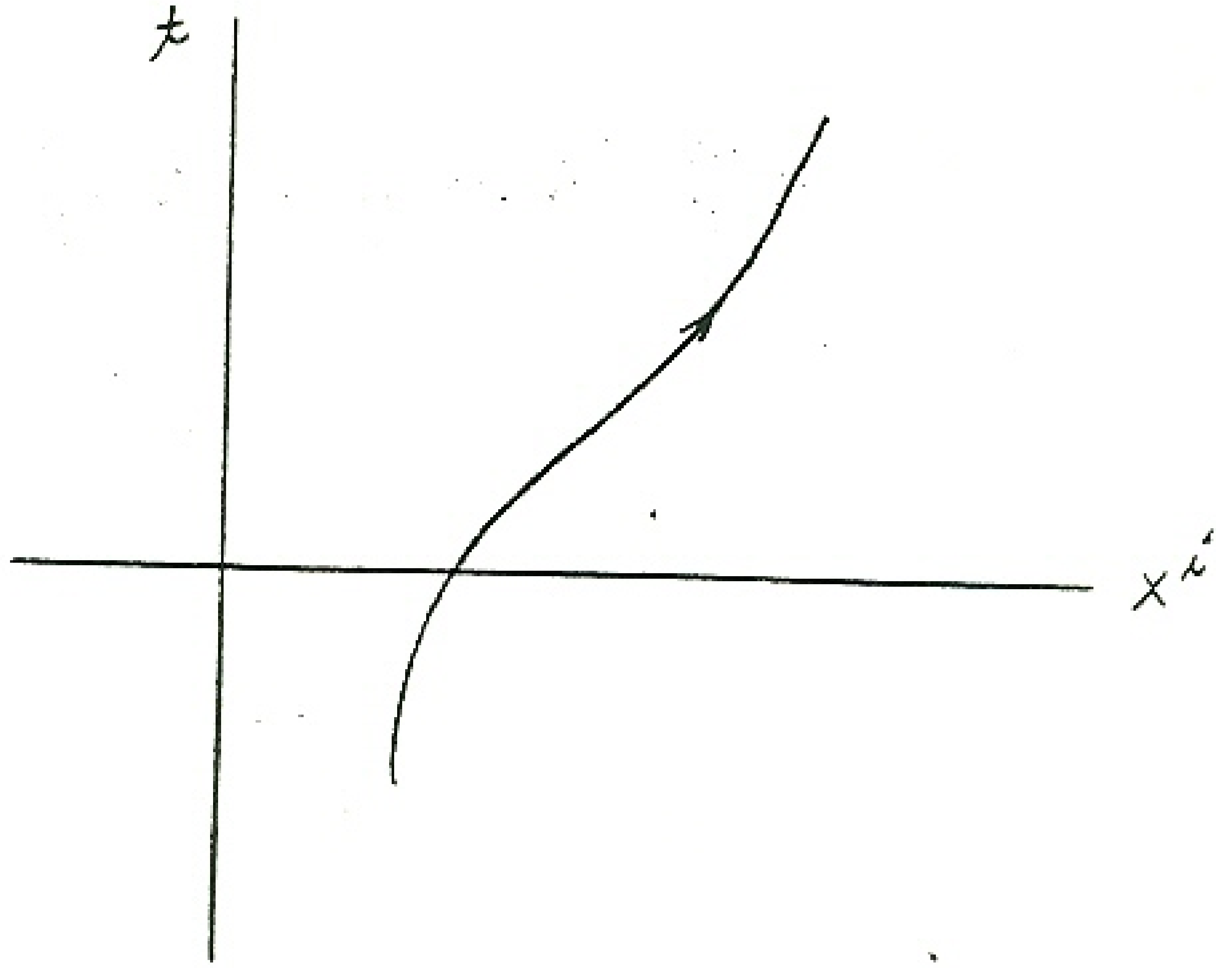


Fig.4

The space-time separation between points on the world line with coordinates $x^\mu$ and $x^\mu + dx^\mu$ is ds. Dividing this by c we obtain

$$d\tau = ds/c = (1/c)[(dx^0)^2 - |d\vec{x}|^2]^{1/2}$$

$$= dt(1 - \vec{v}^2/c^2)^{1/2} \qquad (43)$$

If a coordinate system was attached to the particle so that its spatial coordinates did not change, then $d\vec{x} = 0$, and $d\tau = dt$, so we interpret $d\tau$ as the time interval measured by an clock moving with the particle. By summing all of the $d\tau$'s between points $p_1$ and $p_2$, we obtain

$$\tau_{12} = \int_{p_1}^{p_2} d\tau = \int_{p_1}^{p_2} dt(1 - v^2/c^2)^{1/2} \qquad (44)$$

This is called the <u>proper time</u> interval between the two points, measured along the world line. It is invariant; all observers must agree on its value. However, it does depend on the path connecting the points. A different world line passing through the same two points would give a different value of $\tau_{12}$. We may summarize the distinction between the increment of proper time $d\tau$ and the increment of ordinary time $dt$ as follows:

<u>$d\tau$ is an invariant but not an exact differential; $dt$ is an exact differential but not an invariant.</u>

Presumably, it is the proper time that is recorded by a clock that follows the world line, whether the clock is mechanical, atomic or biological. When the relativistic laws of physics are discussed in the next chapter, we shall find reasons for believing this to be the case. This has led to a prediction that many have found paradoxical; certainly it is startling at first encounter. It is known as the "clock paradox" or "twin paradox". Consider two identical twins, Jim and John born in an inertial system S. One of the twins, Jim, enters a rocket ship and journeys to a star at a distance of 12 light years, traveling with a velocity of 0.6c. His brother John stays home. On reaching the star, Jim quickly turns around and returns home at the same speed. The world lines of the two twins are shown in Fig.5.

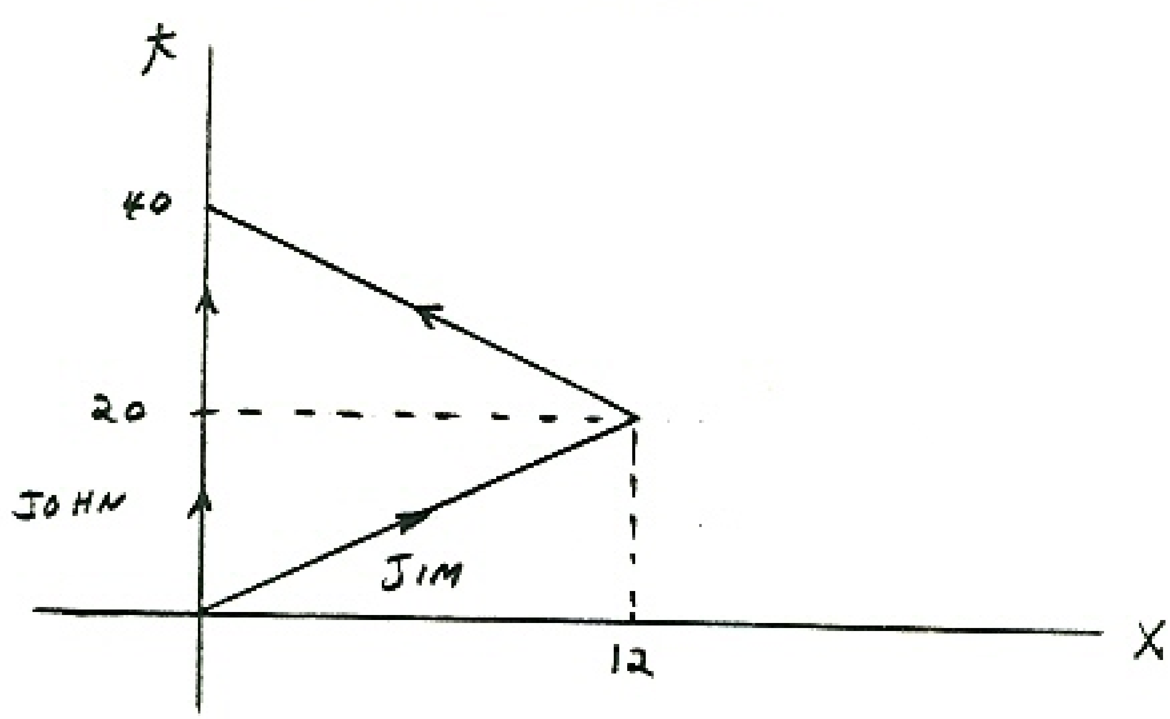

Fig.5.

During the journey John will have aged by

$$\tau(\text{John}) = \int_0^{40} dt \quad = 40 \text{ years} \qquad (45a)$$

while Jim will have aged by

$$\tau(\text{Jim}) = \int_0^{40} dt[1 - (0.6)^2]^{1/2} = (0.8)(40) = 32 \text{ years} \quad (45b)$$

The twins are no longer the same age when they are reunited.

We have done the the calculation in the inertial system in which John is stationary, but that is not important. An observer in any other inertial system would get the same result since $\tau$ is invariant.

It has been argued that this situation is a paradox for the following reason. We could have assumed that it was Jim's rocket ship that was stationary and that it was John who took the long journey with velocity 0.6c. Then there would have been a reversal of roles with $\tau(\text{John})$ = 32 years and $\tau(\text{Jim})$ = 40 years.
It has been argued that either twin has the right to consider himself the stationary one, and hence there is a contradiction. The resolution of the paradox is that Jim does not have the right to consider himself stationary. His rocket ship is not an inertial system. He can determine that it is not by observing the inertial forces that act when he reverses his direction to begin his return trip. It is possible to calculate $\tau$ in Jim's accelerated system, but then $d\tau$ is not given simply by $d\tau^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta / c^2$.
This expression is valid only for inertial systems.

The twin or clock paradox has been a subject of controversy ever since the publication of Einstein's paper on special relativity in 1905, and the literature on the subject is now quite voluminous. A few years ago an experiment was done that tested the prediction in the most direct way possible. A cesium atomic clock was taken on an around-the-world airplane flight. After its return it was compared with a similar clock that remained behind. The two clocks had followed different space-time paths and should have recorded different elapsed times. Although this difference in elapsed times was extremely minute, the accuracy of cesium clocks is sufficiently good that the effect was observable. It agreed with the predictions of the theory of relativity within experimental error.

Since Minkowski space is flat, it is always possible to find inertial systems in which the metric tensor is $\eta_{\alpha\beta}$. However, we are not restricted to inertial systems. We may transform to another set of space-time coordinates $x^{\mu'} = x^{\mu'}(x^{\nu})$ which may be accelerated and rotating relative to the first. The new components of the metric tensor are given by the usual formula

$$g_{\mu'\nu'} = x^{\alpha}_{,\mu'}\, x^{\beta}_{,\nu'}\, \eta_{\alpha\beta} \tag{46}$$

In the next chapter we shall formulate the laws of physics in a generally covariant form, so they will be applicable in accelerated systems as well as inertial systems.

# CHAPTER 8

## RELATIVISTIC ELECTRODYNAMICS AND MECHANICS

In this chapter we shall write Maxwell's equations for the electromagnetic field, the equations of particle mechanics and the equations of fluid mechanics as tensor equations in Minkowski space. We shall begin with Maxwell's equations, since for them only the notation is changed; no modification of the physics is required.

### ELECTRODYNAMICS

In the usual notation of 3-dimensional vectors, Maxwell's equations for the electric field $\vec{E}$ and magnetic field $\vec{B}$ are

$$\nabla\cdot\vec{E} = 4\pi\,\rho \qquad (1a)$$

$$\nabla\cdot\vec{B} = 0 \qquad (1b)$$

$$\nabla\times\vec{E} = -(1/c)\,\partial\vec{B}/\partial t \qquad (1c)$$

$$\nabla\times\vec{B} = 4\pi/c\,\vec{J} + (1/c)\,\partial\vec{E}/\partial t \qquad (1d)$$

Because the divergence of $\vec{B}$ vanishes we can write $\vec{B}$ as the curl of a vector potential $\vec{A}$. Thus

$$\vec{B} = \nabla\times\vec{A} \qquad (2)$$

Substituting this into Eq.(1c) gives

$$\nabla\times[\vec{E} + (1/c)\,\partial\vec{A}/\partial t] = 0 \qquad (3)$$

Since the curl of the quantity in square brackets vanishes, we can write it as the gradient of a scalar potential $-\phi$. Then

$$\vec{E} = -(1/c)\partial \vec{A}/\partial t - \nabla\phi \tag{4}$$

We may form a 4-vector from the scalar potential $\phi$ and the three components of $\vec{A}$. We denote it by $A^\mu$ and take the time-like component $A^0$ to be $\phi$ and the space-like components to be the components of $\vec{A}$. Thus

$$A^\mu = (A^0, \vec{A}) = (\phi, \vec{A}) \tag{5}$$

These are the contravariant components of the 4-vector potential in an inertial system with metric tensor $\eta_{\alpha\beta}$. Its covariant components are

$$A_\mu = \eta_{\mu\nu}A^\nu = (\phi, -\vec{A}) \tag{6}$$

Its 4-dimensional curl is the antisymmetric second rank tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7}$$

Writing out its components

$$F_{01} = \partial A_1/\partial x^0 - \partial A_0/\partial x^1 = -(1/c)\partial A^1/\partial t - \partial\phi/\partial x^1 = E^1 \tag{8}$$

and similarly $F_{02} = E^2$ and $F_{03} = E^3$. Also,

$$F_{12} = \partial A_2/\partial x^1 - \partial A_1/\partial x^2 = -B^3 \tag{9}$$

and similarly $F_{13} = +B^2$ and $F_{23} = -B^1$.
Putting it all together, we obtain

$$F_{\mu\nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & +B^3 & 0 & -B^1 \\ -E^3 & -B^2 & +B^1 & 0 \end{bmatrix} \tag{10}$$

The six components of the 3-dimensional field vectors $\vec{E}$ and $\vec{B}$ go together to form an antisymmetric 4-dimensional tensor F that we call the <u>electromagnetic field tensor</u>. We may raise and lower indices in the usual way using the metric tensor η. Thus

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix} \qquad (11)$$

Note that raising or lowering indices with η is very simple. There is a change of sign if the index moved is 1 , 2 or 3 and no change if it is 0. Thus in going from Eq.(10) to (11) there is one change in sign going from $F_{0i}$ to $F^{0i}$ and two changes in sign in going from $F_{ij}$ to $F^{ij}$ where i,j = 1, 2, 3.

We may take the covariant derivative of $F^{\mu\nu}$ and contract to get $F^{\mu\nu}{}_{;\nu}$, the divergence of F. In a system with metric tensor η, the covariant derivative is just the partial derivative. For μ =0 we get

$$F^{0\nu}{}_{;\nu} = F^{01}{}_{,1} + F^{02}{}_{,2} + F^{03}{}_{,3}$$

$$= -\nabla\cdot\vec{E} = -4\pi\rho \qquad (12a)$$

For μ = 1 we get

$$F^{1\nu}{}_{;\nu} = F^{10}{}_{,0} + F^{12}{}_{,2} + F^{13}{}_{,3}$$

$$= (1/c)\,\partial E^1/\partial t - \partial B^3/\partial x^2 + \partial B^2/\partial x^3 = -(4\pi/c)J^1 \qquad (12b)$$

with similar expressions for $\mu = 2, 3$. Defining the current 4-vector by

$$J^\mu = (J^0, \vec{J}) = (c\rho, \vec{J}) \tag{12c}$$

we may write

$$F^{\mu\nu}{}_{;\nu} = -(4\pi/c)\, J^\mu \tag{12d}$$

From the definition of the electromagnetic field tensor F as the curl of the potential A, it follows that

$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu}$$

$$= (A_{\nu,\mu} - A_{\mu,\nu})_{,\lambda} + (A_{\lambda,\nu} - A_{\nu,\lambda})_{,\mu} + (A_{\mu,\lambda} - A_{\lambda,\mu})_{,\nu} \tag{13}$$

This vanishes identically since the order of partial differentiation can be changed.

As 4-dimensional tensor equations, Maxwell's equations take the form

$$F^{\mu\nu}{}_{;\nu} = -4\pi/cJ^\mu \tag{14a}$$

$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0 \qquad \text{(14b)36}$$

The four equations of Eq.(14a) are the four inhomogenous Maxwell's equations of Eq.(1a) and (1d). There are only four nontrivial Eqs.(14b). These are the ones with all three indices different; namely, $(\mu, \nu, \lambda) = (1,2,3), (0,1,2), (0,1,3), (0,2,3)$.
The first of these gives Eq.(1a) and the other three give Eq.(1c). We have written the derivatives as covariant derivatives, so these are tensor equations and are valid in any coordinate system, not necessarily an inertial system with rectangular axes. However, since Minkowski space is flat, it is always possible to use a coordinate system with metric tensor $\eta$, we will do so when it

is convenient; then, covariant derivatives become partial derivatives, and ; is replaced by , to denote a derivative.

By differentiating and contracting Eq.(14a) we get the equation of current conservation; thus

$$F^{\mu\nu}{}_{;\nu;\mu} = -(4\pi/c)J^{\mu}{}_{;\mu} = 0 \qquad (15)$$

This vanishes since the left hand side is antisymmetric in the superscripts and symmetric in the subscripts. The right hand side is the divergence of the current. It vanishes, indicating that electric charge is neither created nor destroyed.

Problem 1.

Show that under a Lorentz transformation to a reference system moving with velocity $\vec{v}$ , the fields $\vec{E}$ and $\vec{B}$ become $\vec{E}'$ and $\vec{B}'$ where

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma[\vec{E}_{\perp} + (\vec{v} \times \vec{B})/c]$$

$$\vec{B}'_{\perp} = \gamma[\vec{B}_{\perp} - (\vec{v} \times \vec{E})/c]$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2}$$

The subscripts $\parallel$ and $\perp$ denote vectors parallel and perpendicular to the velocity $\vec{v}$.

We define the dual electromagnetic field tensor $^*F$ with components

$$^*F_{\mu\nu} = (1/2\sqrt{|g|})\epsilon_{\mu\nu\alpha\beta}\, F^{\alpha\beta} \qquad (16)$$

where g is the determinant of the metric tensor and $\epsilon_{\mu\nu\alpha\beta}$ is the completely antisymmetric Levi-Civita tensor density. The factor $|g|^{-1/2}$ is necessary to convert the tensor density into a tensor which we then contract with the field tensor to obtain its dual. In a Lorentzian reference system $|g| = |\eta| = 1$. A short calculation gives

$$^*F_{\mu\nu} = \begin{bmatrix} 0 & -B^1 & -B^2 & -B^3 \\ +B^1 & 0 & -E^3 & +E^2 \\ +B^2 & +E^3 & 0 & -E^1 \\ +B^3 & -E^2 & +E^1 & 0 \end{bmatrix} \qquad (17a)$$

The indices may be raised to obtain the contravariant components of the dual field tensor; thus

$$^*F^{\mu\nu} = \begin{bmatrix} 0 & +B^1 & +B^2 & +B^3 \\ -B^1 & 0 & -E^3 & +E^2 \\ -B^2 & +E^3 & 0 & -E^1 \\ -B^3 & -E^2 & +E^1 & 0 \end{bmatrix} \qquad (17b)$$

Note that the dual tensor $^*F$ is obtained from the tensor F by the replacements $\vec{E} \to -\vec{B}$ and $\vec{B} \to \vec{E}$.

The dual of the current vector, denoted by $^*J$, is a third rank tensor with components

$$^*J_{\mu\nu\lambda} = (1/2\sqrt{|g|})\epsilon_{\mu\nu\lambda\gamma}\, J^\gamma \qquad (18a)$$

In a reference system with a Lorentzian metric, the independent components are:

$$\begin{aligned} ^*J_{012} &= +\, J^3 \\ ^*J_{013} &= -\, J^2 \\ ^*J_{023} &= +\, J^1 \\ ^*J_{123} &= -\, J^0 \end{aligned} \qquad (18b)$$

Clearly, $^*J_{\mu\nu\lambda}$ is completely antisymmetric; the other components can be obtained from the above by permuting indices.

Maxwell's equations can be written in terms of the dual tensors $^*F$ and $^*J$. A short calculation shows that they are

$$^*F^{\mu\nu}{}_{;\nu} = 0 \tag{19a}$$

$$^*F_{\mu\nu;\lambda} + {}^*F_{\nu\lambda;\mu} + {}^*F_{\lambda\mu;\nu} = (4\pi/c)\ {}^*J_{\mu\nu\lambda} \tag{19b}$$

Maxwell's equations take a very concise form when expressed in the language of differential forms. We define a potential 1-form

$$\tilde{A} = A_\mu \tilde{d}x^\mu = A_0 c\tilde{d}t + A_1 \tilde{d}x + A_2 \tilde{d}y + A_3 \tilde{d}z \tag{20}$$

Taking the exterior derivative gives

$$\tilde{F} = \tilde{d}\tilde{A} = A_{\nu,\mu}\ \tilde{d}x^\mu{}_\wedge \tilde{d}x^\nu$$

$$= 1/2(A_{\nu,\mu} - A_{\mu,\nu})\ \tilde{d}x^\mu{}_\wedge \tilde{d}x^\nu$$

$$= (1/2)F_{\mu\nu}\ \tilde{d}x^\mu{}_\wedge \tilde{d}x^\nu \tag{21a}$$

This defines $\tilde{F}$ the electromagnetic 2-form.
Writing it out we obtain

$$\tilde{F} = E^1\ c\tilde{d}t_\wedge \tilde{d}x + E^2\ c\tilde{d}t_\wedge \tilde{d}y + E^3\ c\tilde{d}t_\wedge \tilde{d}z$$

$$- B^1\ \tilde{d}y_\wedge \tilde{d}z - B^2\ \tilde{d}z_\wedge \tilde{d}x - B^3\ \tilde{d}x_\wedge \tilde{d}y \tag{21b}$$

The second exterior derivative must vanish identically. Writing it out we obtain

$$\tilde{d}\tilde{F} = \tilde{d}\tilde{d}\tilde{A} = 0$$

$$= 1/2\ F_{\mu\nu,\lambda}\ \tilde{d}x^\lambda\ \tilde{d}x^\mu\ \tilde{d}x^\nu$$

$$= 1/3!\ (F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu})\tilde{d}x^{\kappa}{}_{\wedge}\tilde{d}x^{\mu}{}_{\wedge}\tilde{d}x^{\nu} \qquad (22)$$

The dual electromagnetic 2-form is defined to be

$$^{*}\tilde{F} = 1/2\ {}^{*}F_{\mu\nu}\ \tilde{d}x^{\mu}{}_{\wedge}\tilde{d}x^{\nu}$$

$$= -B^1\ c\tilde{d}t_{\wedge}\tilde{d}x - B^2\ c\tilde{d}t_{\wedge}\tilde{d}y - B^3\ c\tilde{d}t_{\wedge}\tilde{d}z$$

$$- E^1\ \tilde{d}y_{\wedge}\tilde{d}z - E^2\ \tilde{d}z_{\wedge}\tilde{d}x - E^3\ \tilde{d}x_{\wedge}\tilde{d}y \qquad (23)$$

Taking the exterior derivative gives

$$\tilde{d}{}^{*}\tilde{F} = 1/2\ {}^{*}F_{\mu\nu,\lambda}\ \tilde{d}x^{\lambda}{}_{\wedge}\tilde{d}x^{\mu}{}_{\wedge}\tilde{d}x^{\nu}$$

$$= 1/3!\,[{}^{*}F_{\mu\nu,\lambda} + {}^{*}F_{\nu\lambda,\mu} + {}^{*}F_{\lambda\mu,\nu}]\tilde{d}x^{\lambda}{}_{\wedge}\tilde{d}x^{\mu}{}_{\wedge}\tilde{d}x^{\nu}$$

$$= (4\pi/c)\ {}^{*}\tilde{J} \qquad (24)$$

where we have used Eq.(19b). Maxwell's equations may now be written as

$$\tilde{d}{}^{*}\tilde{F} = (4\pi/c)\ {}^{*}\tilde{J} \qquad (25a)$$

$$\tilde{d}\tilde{F} = 0 \qquad (25b)$$

<u>Problem 2.</u>

Show that the scalars $F_{\mu\nu}F^{\mu\nu}$ and ${}^{*}F_{\mu\nu}{}^{*}F^{\mu\nu}$ are equal to $-2(E^2 - B^2)$ and that

$$^{*}F_{\mu\nu}F^{\mu\nu} = 4\vec{E}\cdot\vec{B}$$

Suppose we have a vector potential $A_\mu$ and we add to it the gradient of of a scalar function $\chi(x)$ to obtain another vector potential $A'_\mu$; thus

$$A'_\mu = A_\mu + \chi_{,\mu} \tag{26}$$

This is called a gauge transformation. Using the new potential to calculate the field tensor gives

$$F'_{\mu\nu} = A'_{\nu,\mu} - A'_{\mu,\nu} = A_{\nu,\mu} - A_{\mu,\nu} + \chi_{,\nu,\mu} - \chi_{,\mu,\nu}$$
$$= F_{\mu\nu} \tag{27}$$

Since the order of partial differentiation may be interchanged, the terms in the derivatives of $\chi$ cancel. The electromagnetic field tensor is invariant under gauge transformations. This freedom to change the vector potential by gauge transformations is very useful. Let us derive an equation that relates $A^\mu$ to the current $J^\mu$. Substituting Eq.(7) into (14a) gives

$$\partial_\nu F^{\mu\nu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = -(4\pi/c)\, J^\mu \tag{28a}$$

which may be written as

$$\Box^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = (4\pi/c)\, J^\mu \tag{28b}$$

$\partial_\nu$

where the D'alembertian operator

$$\Box^2 = \partial_\nu \partial^\nu \tag{28c}$$

has been used. Now suppose that $\partial_\nu A^\nu \neq 0$. We may make a gauge transformation to a new potential $A'^\nu$ given by Eq.(26) and require

$$\partial_\nu A'^\nu = \partial_\nu A^\nu + \partial_\nu \partial^\nu \chi = 0 \tag{29a}$$

We must choose $\chi$ to satisfy

$$\Box^2 = -\partial_\nu A^\nu \tag{29b}$$

This equation can be solved for $\chi$ by a method we are about to develope. The new potential satisfies

$$\Box^2 A'^{\mu} = (4\pi/c)\, J^{\mu} \tag{30a}$$

with the subsidiary condition

$$\partial_{\nu} A'^{\nu} = 0 \tag{30b}$$

This subsidiary condition is known as the <u>Lorentz condition</u> or the <u>Lorentz gauge</u>. We still have the freedom of adding to $A^{\mu}$ the gradient of a scalar function $\chi$ that satisfies

$$\Box^2 \chi = 0 \; , \tag{30c}$$

so the potential is still not uniquely determined. Having established the possibility of eliminating the second term on the left in Eq.(28b), we shall drop the prime in Eq.(30a,b) and turn our attention to its solution.

First, we consider the homogenous, or source free, equation obtained by setting $J^{\mu} = 0$. We try a solution of the form

$$A^{\mu}(x) = C^{\mu} \exp(-ikx) \tag{31a}$$

where $C^{\mu}$ is a constant 4-vector and $kx$ is an abbreviation for

$$kx = k_{\nu} x^{\nu} = \omega t - \vec{k}\cdot\vec{x} \tag{31b}$$

and the components of the 4-vector $k^{\nu}$ are

$$k^{\nu} = (\omega/c,\ \vec{k}) \tag{31c}$$

Substituting into Eq.(30) gives

$$(\omega^2/c^2 - |\vec{k}|^2)\, C^{\mu} = 0 \tag{32a}$$

$$k_{\nu} C^{\nu} = 0 \tag{32b}$$

This is a plane wave with wave vector $\vec{k}$ and frequency $\omega = \pm c|\vec{k}|$. According to Eq.(32b), the amplitude 4-vector $C^\mu$ must be orthogonal to the wave 4-vector $k^\nu$.

The inhomogenous equation can be solved by using the 4-dimensional Fourier transform

$$A^\mu(k) = \int d^4x \exp(ikx) A^\mu(x) \qquad (33a)$$

and its inverse

$$A^\mu(x) = (2\pi)^{-4} \int d^4k \exp(-ikx) A^\mu(k) \qquad (33b)$$

We take the Fourier transform of Eq.(30a) by multiplying both sides by exp(ikx) and integrating over all space-time. On the left we integrate by parts twice, thus transfering the differential operator $\Box^2$ from $A^\mu$ to exp(ikx). We obtain

$$-k^2 A^\mu(k) = (4\pi/c) J^\mu(k) \qquad (34a)$$

from which

$$A^\mu(k) = \Delta(k) J^\mu(k) \qquad (34b)$$

where

$$\Delta(k) = -4\pi/ck^2$$

$$= -4\pi/c(\omega^2/c^2 - |\vec{k}|^2) \qquad (34c)$$

The solution is now obtained by taking the inverse transform of Eq.(34a). Since the right hand side has the form of a product of two functions of k, we may use the convolution theorem for Fourier transforms to write the solution as

$$A^\mu(x) = \int d^4x' \Delta(x - x') J^\mu(x') \qquad (35a)$$

where $\Delta(x)$ is the inverse transform of $\Delta(K)$; that is

$$\Delta(x) = -[4\pi/c(2\pi)^4] \int d^4k \exp(-ikx)[\omega^2/c^2 - |\vec{k}|^2]^{-1} \quad (35b)$$

Now, we must do the integrations in Eq.(35b).

We write the 4-dimensional volume element as

$$d^4k = (1/c)d\omega d^3k \quad (36a)$$

We introduce spherical coordinates in $\vec{k}$-space with the $k^3$ axis orientated along the vector $\vec{x}$. Then

$$d^3k = |\vec{k}|^2 \, d|\vec{k}| \, \sin\theta \, d\theta \, d\phi \quad (36b)$$

$$\exp(-ikx) = \exp i[|\vec{k}||\vec{x}|\cos\theta - \omega t] \quad (36c)$$

The integrations over the angles $\phi$ and $\theta$ are easily done and one finds

$$\Delta(x) = \quad (36d)$$

$$-(2\pi^2 i|r)^{-1} \int_{-\infty}^{+\infty} d\omega \int_0^{\infty} KdK\, [\omega^2 - c^2K^2]^{-1} (\exp(iKr) - \exp(-iKr))\exp(-i\omega t)$$

where $r = |\vec{x}|$ and $K = |\vec{k}|$. In the term containing $\exp(-iKr)$ we change the variable from $K$ to $K' = -K$ and then drop the prime and combine it with the first integral to extend the range of integration from $-\infty$ to $+\infty$ instead of 0 to $\infty$. We obtain

$$\Delta(x) = -(2\pi^2 ir)^{-1} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{\infty} KdK\, [\omega^2 - c^2K^2]^{-1} \exp i(Kr - \omega t) \quad (36e)$$

We shall do the $\omega$ integration first. We shall consider it as an integration along the real axis of the complex $\omega$-plane as shown in Fig.1. We may close the path by infinite semicircles in either the upper or lower half plane, the choice being dictated by the requirement that the contribution from the integrations along the semicircles vanish as the radii of the semicircles become infinite. Since the integrand contains $\exp(-i\omega t)$, we see that we must close the path in the upper half plane for $t < 0$ and in the lower half plane for $t > 0$. Then we can use the residue theorem to evaluate the integral.

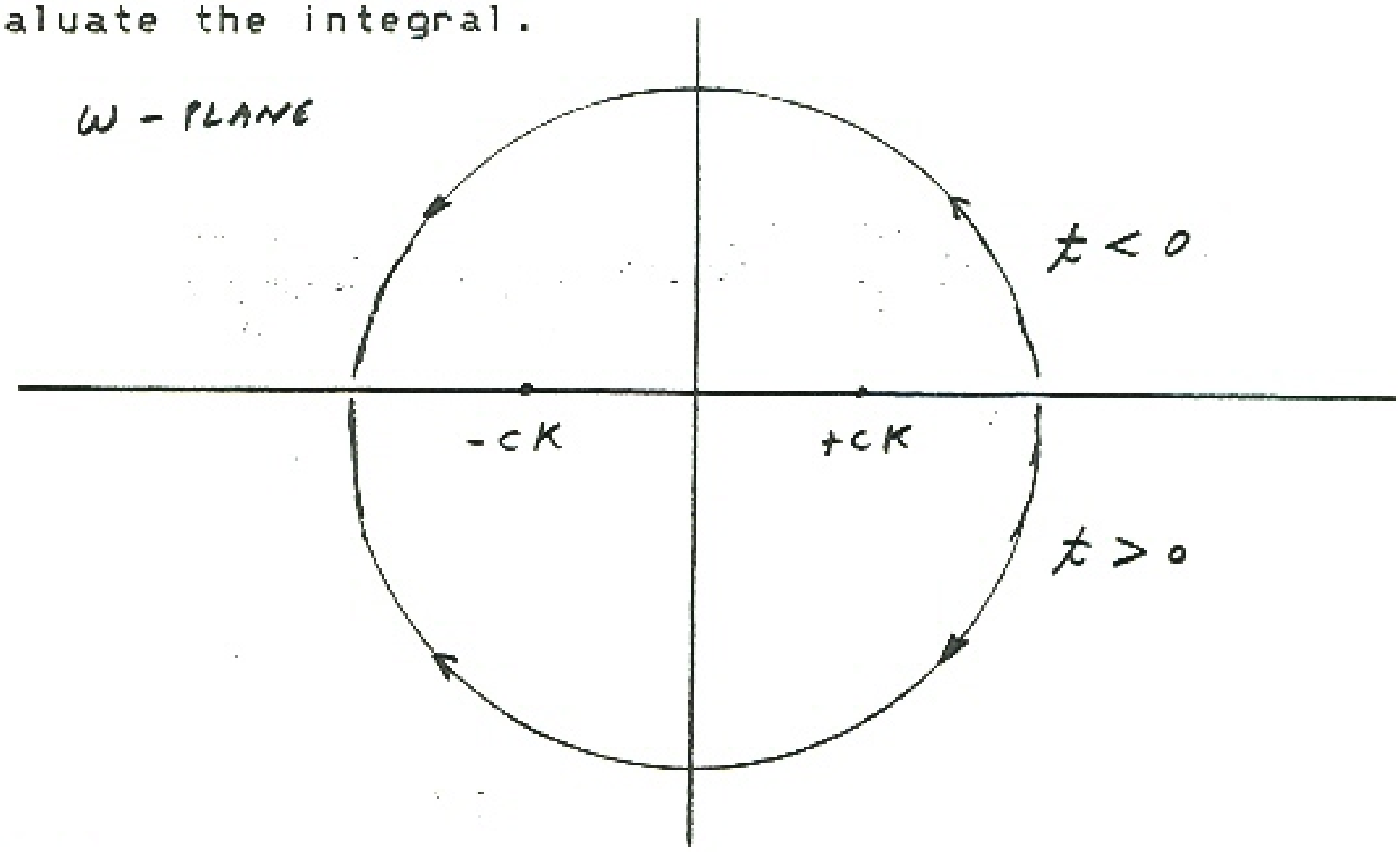


Fig.1.

Now we encounter a problem. The integrand has poles at $\omega = \pm Kc$. These lie on the path of integration. We could distort the path to go above or below one or the other or both poles, or we could take a principle value. Our choice must be dictated by physical considerations. For the moment we shall choose to take the principal value, and then later we shall discuss other choices. When we take the principal value, each pole contributes $i\pi$ times the residue at that pole and we obtain

$$P\int_{-\infty}^{+\infty} d\omega \exp(-i\omega t)\,[(\omega + cK)(\omega - cK)]^{-1}$$

$$= \pm(i\pi/2cK)[\exp(-icKt) - \exp(icKt)] \qquad (37)$$

where the positive sign is to be taken for $t < 0$ and the negative sign for $t$ ~~<~~ > 0. Substituting this into Eq.(36e) gives

$$\Delta(x) = -(\pm)/4\pi cr \int_{-\infty}^{+\infty} dK\,(\exp i(r - ct)K - \exp i(r + ct)K)$$

$$= -(\pm)(1/2cr)[\delta(r - ct) - \delta(r + ct)] \qquad (38a)$$

Since r is always positive, this is

$$\Delta(x) = \Delta(\vec{x}, t) = \begin{cases} (1/2cr)\ \delta(r - ct) \text{ for } t > 0 \\ (1/2cr)\ \delta(r + ct) \text{ for } t < 0 \end{cases} \qquad (38b)$$

Which may be written as

$$\Delta(x) = (1/2cr)[\delta(r - ct) + \delta(r + ct)]$$

$$= (1/c)\delta(r^2 - c^2t^2)$$

$$= (1/c)\delta(x^\nu x_\nu) \qquad (38c)$$

In obtaining this last form we have the used the formula

$$\delta(f(r)) = \sum_i |df/dr|^{-1}\delta(r - r_i) \qquad (39)$$

with $f(r) = r^2 - c^2t^2$, and $r_i = \pm ct$ are the roots of $f(r) = 0$.

From Eq.(35a) we see that $\Delta(x)$ represents the response of the potential to a $\delta$-function current pulse at $x' = (ct', \vec{x}') = 0$. From Eq.(38c) we see that this consist of two parts. One is a spherical wave expanding into the future light cone, and the other is a spherical wave expanding into the past light cone. An observer would interpret this as a converging spherical wave coming in from $r = \infty$ arriving at the origin at $t = 0$ and becoming an expanding spherical wave. This is not what we expect. All of our experience with the real world teaches us that there should be no response prior to the cause. It seems that we have found a physically unreasonable $\Delta(x)$. We should require $\Delta(x) = 0$ for $t < 0$. The trouble was in the way we handled the singularities on the path of integration in Fig.1. We should not have taken the princpal value in Eq.(37). To get a vanishing $\Delta(x)$ for $t < 0$, we could displace the contour an infinitesimal distance above the real axis. Then when we closed the contour in the upper half plane, no poles would be enclosed and the $\omega$ integration would give zero. When we closed the contour in the lower half plane for $t > 0$, both poles would be enclosed and we would get $2\pi i$ (instead of $\pi i$ when we took the principal value) times the sum of the residues. The result would be that we would omit the term $\delta(r + ct)$ in Eq.(38c) and multiply by 2 to obtain

$$\Delta_r(x) = (1/cr)\delta(r - ct) \qquad (40a)$$

We shall call this the retarded Green's function and distinguish it by a subscript r. If we had displaced the contour an infinitesimal distance below the real axis, then we would have obtained no response for $t > 0$, and the Green's function would be

$$\Delta_a(x) = (1/cr)\delta(r + ct) \qquad (40b)$$

We shall call this the advanced Green's function and distinguish it by a subscript a. With this Green's function the response preceeds the cause. The first Green's function, that we found by taking the principal value, Eq.38c), is the average of these two. We shall call it the Wheeler-Feynman Green's function and denote it by $\Delta_{WF}(x)$. It plays a central role in a theory called action-at-a-distance electrodynamics that is discussed in the next chapter.

The requirement that the response follow the cause compells us to choose the retarded Green's functionan and write the potential as

$$A^\mu(\vec{x}, t) = \int dt' d^3x' \frac{\delta[|\vec{x} - \vec{x}'| - c(t-t')]J^\mu(\vec{x}', t')}{|\vec{x} - \vec{x}'|} \qquad (41)$$

Next, we would like to use this solution for the potential to calculate the electromagnetic field due to a collection of arbitrarily moving point particles. We will label each particle with a subscript a and let its charge be $e_a$ and its position at time t be $\vec{a}(t)$. The charge and current densities associated with particle a are

$$\rho_a(\vec{x},t) = e_a\, \delta(\vec{x} - \vec{a}(t)) \qquad (42a)$$

$$\vec{J}_a(\vec{x},t) = e_a(d\vec{a}/dt)\, \delta(\vec{x} - \vec{a}) \qquad (42b)$$

We define a 4-velocity for particle a whose space-time coordinates are $a^\mu$ by

$$u^\mu{}_a = da^\mu/da \qquad (43a)$$

where we have denoted the differential of proper time along the path of particle a by

$$da = (1/c)(\eta_{\alpha\beta}da^\alpha da^\beta)^{1/2} = dt_a(1 - (d\vec{a}/cdt)^2)^{1/2} \qquad (43b)$$

We shall use a dot over a symbol to denote the derivative with respect to proper time. Thus the 4-velocity of particle a is

$$\dot{a}^\mu = da^\mu/da = (c,\ d\vec{a}/dt)\ dt/da \qquad (44)$$

These may be used to write the 4-vector current of particle a as

$$J^\mu_a = e_a\ \dot{a}^\mu \delta(\vec{x} - \vec{a}(t))\ da/dt \qquad (45)$$

We find it convenient to use Eq.(38c) for the Green's function; then to get the retarded response we take only the contribution from the past light cone of point x and multiply by 2. Substituting Eq.(45) and (38c) into (35a) we obtain

$$A^\mu_a(\vec{x},t) = e_a\int dt' d^3x'\ \delta[|\vec{x} - \vec{x}'| - c^2(t-t')^2]\ \dot{a}^\mu \delta[\vec{x}' - \vec{a}(t')]\ da/dt'$$

$$= e_a\int da\ \delta(xa\cdot xa)\dot{a}^\mu. \qquad (46)$$

In this last equation we have denoted the space-time vector connecting point a to point x by xa; its components are $x^\nu - a^\nu$.
The dot between xa and xa denotes the scalar product

$$xa\cdot xa = (x^{\nu} - a^{\nu})(x_{\nu} - a_{\nu}) \qquad (47)$$

In Fig. 2 we have sketched the world line of particle a and the light cone from point x with its intersections with the world line. We remind the reader that in evaluating Eq.(46) we take only the contribution from the past light cone and multiply by 2 to get the retarded fields.

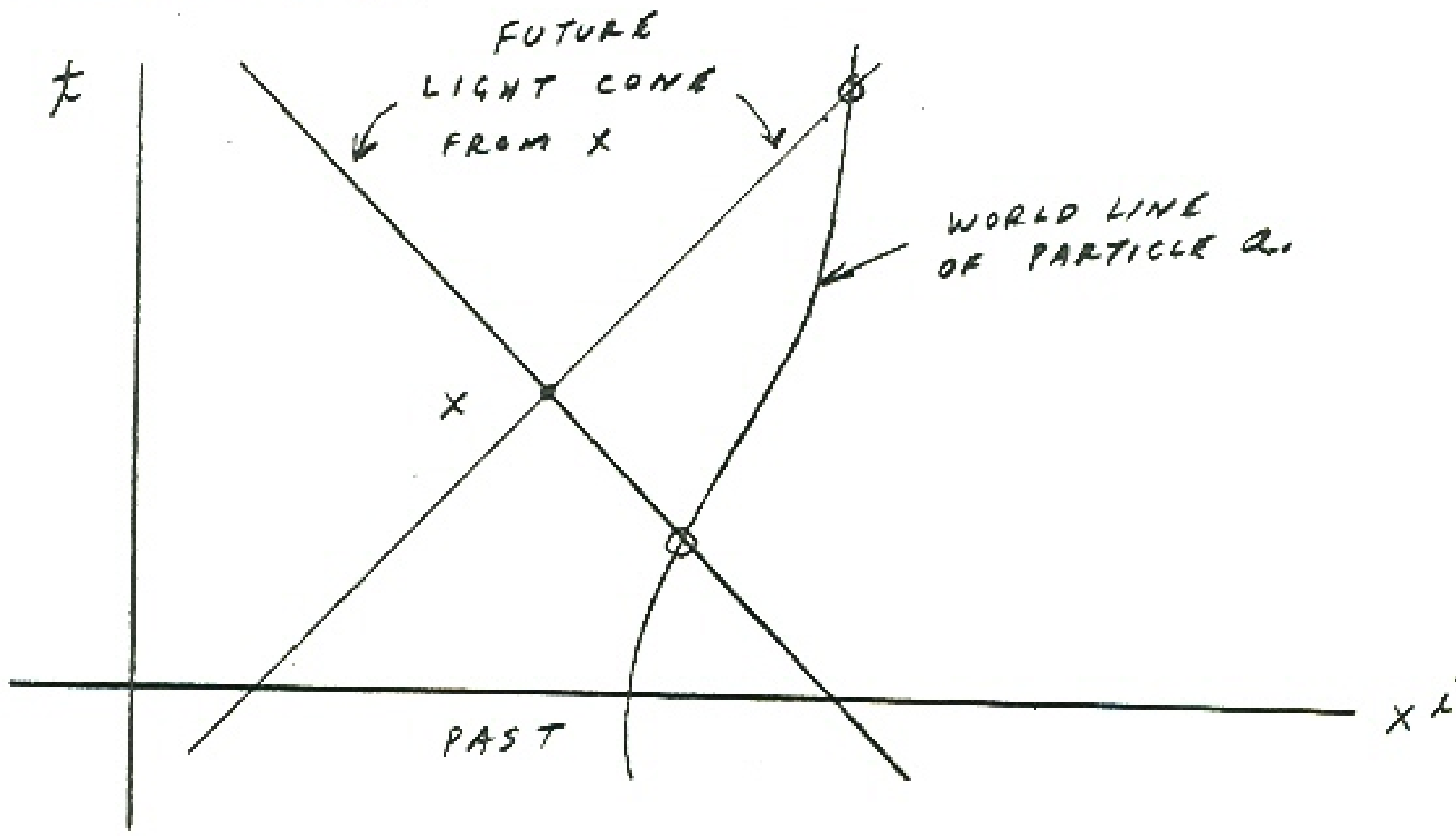


Fig.2.

We may use the δ-function identity of Eq.(39) to evaluate the integral in Eq.(46) to obtain

$$A^{\mu}_{a}(x) = e_a\left[\frac{\dot{a}^{\mu}}{|\dot{a}\cdot xa|}\right]_{ret} \qquad (48)$$

The subscript ret indicates that all quantities are to be evaluated at the retarded time $t' = t - |\vec{x} - \vec{x}'|/c$, and $\dot{a}\cdot xa$ is an abbreviation for $\dot{a}^{\nu}(x - a)_{\nu}$. This is known as the <u>Lienard-Weichert potential</u>.

We can now calculate the electromagnetic field tensor from Eq.(7). It is somewhat easier not to use Eq.(48) but to go back to Eq.(46) and differentiate under the integral sign obtaining

$$F_a^{\mu\nu} = e_a \int da(\dot{a}^\nu \partial^\mu - \dot{a}^\mu \partial^\nu)\delta(f) \qquad (49a)$$

where $f = xa \cdot xa$. It is convenient to take $f$ to be the variable of integration and write

$$F_a^{\mu\nu} = e_a \int df\ (da/df)(\dot{a}^\nu \partial^\mu f - \dot{a}^\mu \partial^\nu f) d\delta(f)/df \qquad (49b)$$

Integration by parts gives

$$F_a^{\mu\nu} = -\, e_a \int df\ \delta(f)\ (da/df)(d/da)[(da/df)(\dot{a}^\nu \partial^\mu f - \dot{a}^\mu \partial^\nu f)]$$

$$= -\, e_a\ ((da/df)(d/da)[(da/df)(\dot{a}^\nu \partial^\mu f - \dot{a}^\mu \partial^\nu f]))_{f=0} \qquad (49c)$$

The quantities necessary for the evaluation of Eq.(49c) are

$$\partial^\mu f = 2(x^\mu - a^\mu) = 2(xa)^\mu \qquad (50a)$$

$$df/da = (da/df)^{-1} = -\,2(xa)^\lambda \dot{a}_\lambda \qquad (50b)$$

Using these, taking the contribution from only the past light cone and multiplying by 2, Eq.(49c) becomes

$$F_a^{\mu\nu} = -\, e_a \left\{ \frac{1}{(xa)^\lambda \dot{a}_\lambda} \frac{d}{da} \frac{\dot{a}^\nu (xa)^\mu - \dot{a}^\mu (xa)^\nu}{(xa)^\lambda \dot{a}_\lambda} \right\}_{ret} \qquad (51)$$

Some straightforward but tedious algebra is necessary to put this into a form that is useful for calculations. The results are

$$\vec{E}(\vec{x},t) = e_a \left\{ \frac{(\vec{n} - \vec{\beta})(1 - \beta^2)}{R^2\ \kappa^3} + \frac{[\vec{n} \times ((\vec{n} - \vec{\beta}) \times d\vec{\beta}/dt)]}{cR\ \kappa^3} \right\}_{ret} \qquad (52a)$$

$$\vec{B}(\vec{x},t) = [\vec{n} \times \vec{E}]_{ret} \qquad (52b)$$

where

$$\vec{n} = (\vec{x} - \vec{a})/R \qquad (52c)$$

$$R = |\vec{x} - \vec{a}| \qquad (52d)$$

$$\vec{\beta} = (1/c)\ d\vec{a}/dt \qquad (52e)$$

These are the fields due to the particle we have labeled a. To find the fields due to a number of particles, we simply sum over a. This completely solves the problem of finding the fields due to a collection of arbitrarily moving particles.

Note that in Eq.(52a) the first term decreases with distance from the particle as $R^{-2}$ while the second term is proportional to the acceleration and decreases as $R^{-1}$. At large distances the second term is dominant if the particle is accelerated. It represents the radiation emitted by an accelerated particle. We shall derive a useful formula for the power in the radiation emitted by an accelerated particle. In our derivation we shall assume that the particle is moving slowly ($\beta \ll 1$), but later we shall generalize our result so that it is applicable to arbitrarily moving particles. At large distances from the particle we need keep only the radiation terms in the fields and obtain

$$\vec{E} = (e/cR)\ \vec{n} \times (\vec{n} \times d\vec{\beta}/dt) \qquad (53a)$$

$$\vec{B} = \vec{n} \times \vec{E} \qquad (53b)$$

The flux of energy is given by the Poynting vector

$$\vec{S} = (\vec{E} \times \vec{B})c/4\pi$$

$$= (c/4\pi)|\vec{E}|^2\vec{n}$$

$$= (e^2/4\pi cR^2)|d\vec{\beta}/dt|^2(1 - \cos^2\theta)\vec{n} \qquad (53c)$$

where $\theta$ is the angle between $\vec{n}$ and $d\vec{\beta}/dt$. This flux of energy is radially outward from the particle and decreases as $R^{-2}$. Integrating this flux over a sphere of radius R centered on the particle, we find that the energy radiated per unit time is

$$P = (2e^2/3c^3)\ |d^2\vec{a}/dt^2|^2 \qquad (54)$$

This useful formula is known as Larmor's formula.

We now ask what the relativistic genrealization of this formula must be. We could go back to Eq.(52) for the fields and carry out a similar calculation without the assumption $\beta \ll 1$, but this is a difficult calculation. Instead we shall try to construct a relativistic scalar that reduces to Eq.(54) in the nonrelativistic limit $\beta \ll 1$. The only quantities available for the construction are the particles 4-velocity $u^\nu$ and its 4-acceleration $du^\nu/da$. From these we can construct the scalars

$$u^\nu u_\nu = \gamma^2\ (c^2 - v^2) = c^2 \qquad (55a)$$

$$(du^\nu/da)(du_\nu/da) \qquad (55b)$$

$$u^\nu(du_\nu/da) \qquad (55c)$$

The first of these is a constant, so it need not be considered. The last may be shown to vanish by differentiating Eq.(55a), so it need not be considered. This leaves Eq.(55b) as the only choice, and in the nonrelativistic limit it reduces to the square of the nonrelativistic acceleration. We are led to write the relativistic generalization of the Larmor formula as

$$P = (2e^2/3c^3)|(d^2a^\mu/da^2)(d^2a_\mu/da^2)| \qquad (56)$$

Problem 3.

Show that Eq.(56) may be written as

$$P = (2e^2/3c^3)\ \gamma^6\ (\ |d\vec{\beta}/dt|^2 - |\vec{\beta} \times d\vec{\beta}/dt|^2)$$

where

$$\gamma = (1 - \beta^2)^{-1/2}$$

PARTICLE MECHANICS

Unlike electrodynamics, mechanics must be modified before it is in accord with the postulates of special relativity. In making this modification we shall be guided by our aim of expressing physical laws as relations among 4-tenaors. When expressed this way, the laws are manifestly covariant with respect to space-time coordinate transformations. Also we expect the modified equations to reduce to the equations of Newton's mechanics in the limit that the velocity is much smaller than the velocity of light, since we know that in this limit Newtonian mechanics is an excellent approximation.

We ask: how may Newton's second law relating force and acceleration

$$m d\vec{v}/dt = \vec{F} \tag{57}$$

be modified to make it a relation among 4-tensors? An obvious choice is to replace the 3-vectors of Eq.(57) by 4-vectors. We may construct a 4-vector velocity by dividing the infinitesimal displacements of the position vector of a particle $dx^\mu$ by the increment of proper time $d\tau$ along the world line of the particle to obtain

$$u^\mu = dx^\mu/d\tau \tag{58a}$$

where

$$d\tau = 1/c\ (\eta_{\alpha\beta} dx^\alpha dx^\beta)^{1/2} = dt(1 - v^2/c^2)^{1/2} \tag{58b}$$

The contravariant and covariant components of the 4-velocity are seen to be

$$u^{\mu} = \gamma(c, \vec{v}) \tag{58c}$$

$$u_{\mu} = \gamma(c, -\vec{v}) \tag{58d}$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2} \tag{58e}$$

We write the generalization of Newton's second law as

$$m du^{\mu}/d\tau = F^{\mu} \tag{59}$$

To use this equation we must be able to express the force acting on the particle as a 4-vector. Let us consider the electromagnet force on a particle of charge e. It is given by

$$\vec{F} = e[\vec{E} + (\vec{v} \times \vec{B})/c] \tag{60}$$

We wish to construct a 4-vector whose spatial components reduce to this in the limit of $v \ll c$. Now $\vec{E}$ and $\vec{B}$ are contained in the electromagnetic field tensor $F^{\mu\nu}$ and the velocity $\vec{v}$ is contained in the 4-velocity $u_{\nu}$.
We can obtain a 4-vector by contracting these two tensors. Thus we are lead to try

$$F^{\mu} = e/c\ F^{\mu\nu} u_{\nu} \tag{61a}$$

For $\mu = 1$ this is

$$F^1 = e/c\ (F^{10}u_0 + F^{12}u_2 + F^{13}u_3$$
$$= e\gamma[\vec{E} + (\vec{v} \times \vec{B})/c]^1 \tag{62}$$

This does indeed reduce to the first component of Eq.(60) in the limit $v \ll c$. We are lead to write the relativistic equations of motion for a charged particle in an electromagnetic field as

$$m du^\mu/d\tau = m d^2x^\mu/d\tau^2 = e/c\ F^{\mu\nu}u_\nu = e/c\ F^\mu{}_\nu dx^\nu/d\tau \quad (63)$$

The three spatial components of this equation are

$$\gamma d/dt(m\gamma\vec{v}) = e\gamma[\vec{E} + (\vec{v} \times \vec{B})/c] \quad (64)$$

A factor $\gamma$ can be ~~factored~~ CANCELLED from both sides; then this equation differs from the nonrelativistic equation of motion by the factor $\gamma$ inside the parenthesis, so it does indeed reduce to the non-relativistic equation in the limit $v << c$. For $\mu = 0$ we obtain

$$d/dt(\gamma mc^2) = e\vec{E}\cdot\vec{v} \quad (65)$$

where we have canceled a factor of $\gamma$ from both sides and multiplied both sides by c. The right hand side is the rate at which the electromagnetic field does work on the particle. This leads us to identify $\gamma mc^2 = E$ as the energy of the particle. For small values of the velocity this becomes

$$E = mc^2(1 - v^2/c^2)^{-1/2} \approx mc^2 + mv^2/2 \quad (66)$$

The second term is the nonrelativistic kinetic energy of the particle. The first term $mc^2$ is present even when the particle is at rest and is called the rest energy of the particle.

We define a momentum 4-vector by

$$p^\mu = mu^\mu = \gamma(mc, m\vec{v}) = (E/c, \vec{p}) \quad (67)$$

Note that the spatial components of the 4-momentum differ from the nonrelativistic 3-momentum by the factor $\gamma$ and that the time component is the particles energy divided by c.

The presence of $d\tau$ in the denominator of both sides of Eq.(63) makes it possible to write the equations in terms of an arbitrary parameter $\lambda$. Let the world line of the particle be given by $x^\mu = x^\mu(\lambda)$. Then the proper time is $\tau = \tau(\lambda)$. We shall use a dot over a symbol to denote a derivative with respect to $\lambda$. Thus

$$u^\mu = dx^\mu/d\tau = \dot{x}^\mu \, d\lambda/d\tau \tag{68a}$$

$$d\tau/d\lambda = 1/c \, (\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)^{1/2} \tag{68b}$$

Using these in Eq.(63) and canceling a factor $d\lambda/d\tau$ from both sides gives

$$d/d\lambda \, [m\dot{x}^\mu \, d\lambda/d\tau] = e/c \, F^\mu{}_\nu dx^\nu/d\lambda \tag{69a}$$

Lowering the superscript $\mu$ gives

$$\frac{d}{d\lambda}\left[\frac{mc \, \eta_{\mu\beta}\dot{x}^\beta}{(\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)^{1/2}}\right] = e/c \, (\partial_\mu A_\nu - \partial_\nu A_\mu)\dot{x}^\nu \tag{69b}$$

It is easily seen that these equations are the Lagrange equations

$$d/d\lambda \, (\partial L/\partial \dot{x}^\mu) - \partial L/\partial x^\mu \tag{70a}$$

when the Lagrangian is taken to be

$$L(x, \dot{x}) = -mc(\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)^{1/2} - e/c \, A_\nu \dot{x}^\nu \tag{70b}$$

We have chosen the sign so as to agree with the usual nonrelativistic Lagrangian in the $v \ll c$ limit.

If we choose the parameter $\lambda$ to be the time t and use $A_\nu = (\phi, -\vec{A})$, Eq.(70) becomes

$$L = -mc^2(1 - v^2/c^2)^{1/2} - e\phi + e/c \, \vec{v}\cdot\vec{A} \tag{71a}$$

In the $v \ll c$ limit this is

$$L = -mc^2 + mv^2/2 - e\phi + e/c \, \vec{v}\cdot\vec{A} \tag{71b}$$

This differs from the usual nonrelativistic Lagrangian only by the constant term $- mc^2$.

The action S is defined to be

$$S = \int L\,dt \quad = \int L\,d\lambda \quad \text{IN GENERAL} \tag{72a}$$

and the Lagranges equations are derived from the requirement that the path followed by the particle be the path that makes S a minimum; that is, $\delta S = 0$ for all variations of the path $\delta x^\mu(\lambda)$.
We write the functional derivative of S as

$$\delta S/\delta x^\mu(\lambda) = \partial L/\partial x^\mu - d/d\lambda\,(\partial L/\partial \dot{x}^\mu) \tag{72b}$$

Using Eqs.(70b) and (68b) we can write the action as

$$S = - mc^2 \int d\tau \; -e/c \int A_\nu dx^\nu \tag{73}$$

We may also find a Hamiltonian and write the equations of motion in Hamiltonian form. We take $\lambda = t$ and use Eq. (71a) for the Lagrangian. The momenta are

$$p_i = \partial L/\dot{x}^i = \gamma m v^i + e/c\, A^i \tag{74a}$$

for $i = 1,2,3$. The Hamiltonian is defined to be

$$H = p_i \dot{x}^i - L$$

$$= mc^2\gamma + e\Phi \tag{74b}$$

We can solve Eq.(74a) for $\vec{v}$ and then calculate

$$\gamma = (1 - v^2/c^2)^{-1/2} = (1/mc^2)[c^2|\vec{p} - e/c\vec{A}|^2 + m^2c^4]^{1/2} \tag{74c}$$

Using this in Eq.(74b) gives the Hamiltonian

$$H = [c^2|\vec{p} - e/c\vec{A}|^2 + m^2c^4]^{1/2} + e\Phi \tag{74d}$$

One may easily check that the Hamiltonian equations

$$\dot{x}^i = \partial H/\partial p_i \tag{75a}$$

$$\dot{p}_i = -\partial H/\partial x^i \tag{75b}$$

give the same equations of motion that were previously found.

In the work of this section up to this point we have assumed that our coordinate system was an inertial system with a Lorentzian metric tensor $\eta_{\alpha\beta} = \mathrm{diag}(+1,-1,-1,-1)$. Since the space-time of special relativity is flat, it is always possible to choose such a coordinate system, but is not necessary. For a general coordinate system with metric tensor $g_{\alpha\beta}(x)$, the Lagrangian of Eq.(70a) is replaced by

$$L = -mc(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{1/2} - e/c\, A_\nu \dot{x}^\nu \tag{76a}$$

We may use the results of Chapter 3 (see Eqs.(66) to (73) of that chapter) to write Lagrange's equations as

$$m[d^2x^\mu/d\tau^2 + \Gamma^\mu_{\alpha\beta}(dx^\alpha/d\tau)(dx^\beta/d\tau)] = e/c\, F^\mu{}_\nu(dx^\nu/d\tau) \tag{76b}$$

These are tensor equations, so if they are valid in one coordinate system they are valid in all coordinate systems. In a coordinate system with $g_{\alpha\beta} = \eta_{\alpha\beta}$, $\Gamma^\mu_{\alpha\beta} = 0$, and Eq.(76b) reduces to our previously found equations of motion, Eq.(63).

One may easily show that the same equations of motion are found from the Lagrangian

$$L = -m/2\, g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - e/c\, A_\nu\dot{x}^\nu \tag{77}$$

where the dot over a symbol denotes a derivative with respect to the proper time $\tau$. The canonical momenta are

$$p_\mu = \partial L/\partial \dot{x}^\mu = -mg_{\mu\nu}\dot{x}^\nu - e/c\, A_\mu \tag{78a}$$

The Hamiltonian is

$$H = p_\mu \dot{x}^\mu - L$$
$$= - m/2\ g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$
$$= - 1/2m\ g^{\alpha\beta}(p_\alpha - e/c\ A_\alpha)(p_\beta - e/c\ A_\beta) \qquad (78b)$$

As the final topic of this section we shall write an action function for a collection of charged particles and an electromagnetic field and show that both the field equations and the equations of motion of the particles follow from the requirement that the action be an extremum. We denote general space-time coordinates by $x^\mu$ and the coordinates of the particle labeled a by $a^\mu$. The metric tensor is $g_{\mu\nu}(x)$. We denote its determinant by g, and since the signature of space-time is (+1, -3), $g = -|g|$. The differential of proper time along the world line of particle a is denoted by $da = (g_{\alpha\beta}da^\alpha da^\beta)^{1/2}/c$. The action functional is

$$S(a^\mu, A_\alpha) = - \sum_a m_a c^2 \int da - \sum_a e_a/c \int A_\mu \dot{a}^\mu da$$
$$- (1/16\pi c) \int d^4x(-g)^{1/2}\ F_{\mu\nu}F^{\mu\nu}$$
$$= S_p + S_{pf} + S_f \qquad (79)$$

S is to be regarded as a functional of the world lines of all of the particles $a^\mu(a)$, and the potentials $A^\mu(x)$. We have divided S into three parts. $S_p$, the first term, contains only particle coordinates $a^\mu$ and their derivatives $\dot{a}^\mu$. $S_f$, the third terms contains only the fields $A_\mu$ and their derivatives $F_{\mu\nu}$. $S_{pf}$ contains both $A_\mu$ and $\dot{a}^\mu$ and represents the interaction of particles and fields.

The particle trajectories and field configurations that make S an extremum are determined by setting the functional derivatives $\delta S/\delta a^\mu$ and $\delta S/\delta A_\nu$ equal to zero. To find the particle trajectories, we may write

$$(\delta/\delta a^{\mu})(S_p + S_{pf}) = \delta S_a/\delta a^{\mu} \qquad (80a)$$

where

$$S_a = - m_a c^2 \int da \; - e_a/c \int A_{\nu} da^{\nu} \qquad (80b)$$

This is the action of Eq.(73) applied to particle a. We know from Eq.(76b) that the equations for the trajectory are

$$m_a [d^2 a^{\mu}/da^2 + \Gamma^{\mu}_{\alpha\beta}(da^{\alpha}/da)(da^{\beta}/da)]$$

$$= e/c \; F^{\mu}{}_{\nu}(da^{\nu}/da) \qquad (80c)$$

To find the functional derivative of $S_f$ we write

$$S_f = \int d^4x \; L_f \qquad (81a)$$

where

$$L_f = - (1/16\pi c)(-g)^{1/2} F_{\mu\nu} F^{\mu\nu} \qquad (81b)$$

Then

$$\delta S_f/\delta A_{\alpha} = \partial L_f/\partial A_{\alpha} - (\partial/\partial x^{\beta})(\partial L_f/\partial A_{\alpha,\beta})$$

$$= - (1/8\pi c)(\partial/\partial x^{\beta})[(-g)^{1/2} F^{\mu\nu} \partial F_{\mu\nu}/\partial A_{\alpha.\beta}]$$

$$= -(1/4\pi c)(\partial/\partial x^{\beta})[(-g)^{1/2} \; F^{\beta\alpha}] \qquad (81c)$$

By writing the current due to particle a as

$$J^{\mu}_a = e_a (-g)^{-1/2} \delta^3(x^i - a^i) \; \dot{a}^{\mu} \; da/dt \qquad (82a)$$

we can write

$$S_{pf} = - \sum_a (1/c^2) \int d^4x (-g)^{1/2} A_{\mu} J^{\mu}_a \qquad (82b)$$

and

$$\delta S_{pf}/\delta A_\alpha = -(1/c^2)(-g)^{1/2} J^\alpha \qquad (82c)$$

where

$$J^\alpha = \sum_a J_a^\alpha \qquad (82d)$$

Setting the sum of Eqs.(81c) and (82c) equal to zero gives

$$(-g)^{-1/2}(\partial/\partial x^\beta)[(-g)^{1/2}F^{\beta\alpha}] = 4\pi/c \ J^\alpha \qquad (83a)$$

which may be written as

$$F^{\alpha\beta}{}_{;\beta} = -4\pi/c \ J^\alpha \qquad (83b)$$

These equations are recognized as the inhomogeneous Maxwell's equations. The homogenous equations follow as identities from the definition of $F_{\mu\nu}$ as the 4-dimensional curl of $A_\mu$.

We see that both Maxwell's equations and the equations of motion of particles follow from $\delta S = 0$ with S given by Eq.(79). This is a very concise statement of the major laws of physics and is independent of the choice of coordinate system. Eqs.(80c) and (83b) are fully covariant.

Problem 4.

Solve Eq.(63) for a particle in a uniform and constant electric field in the z-direction. Show that the particles velocity approaches c asymptoticly from below.

problem 5.

Solve Eq.(63) for a particle in a uniform and constant magnetic field in the z-direction. Show that

$$u^1 = A \sin(\omega_c \tau + \delta)$$

$$u^2 = A\cos(\omega_c\tau + \delta)$$

$$u^3 = \text{constant.}$$

$$u^0 = \text{constant.}$$

where $\omega_c = eB/mc$ is the cyclotron frequency. Note that the period is $2\pi\gamma/\omega_c$, when measured in terms of t instead of $\tau$, so it depends on the velocity of the particle.

Problem 6.

Consider an electron of charge $-|e|$ moveng in the Coulomb field of a nucleus of charge +Ze fixed at the origin. Assume that the motion is in a plane and use polar coordinates r and $\theta$. Write Eq.(77) as

$$L = -(mc^2/2)\dot{t}^2 + m/2(\dot{r}^2 + r^2\dot{\theta}^2) - (Ze^2/r)\dot{t} \qquad mc^2$$

Show that $p_t$ and $p_\theta$, the momenta conjugate to t and $\theta$, and the Hamiltonian H are constants of the motion. Change the independent variable from $\tau$ to $\theta$ and use the constants of the motion to find a first order equation for $u = 1/r$ as a function of $\theta$. Show that a solution is

$$u = 1/r = \frac{1 + e\cos\omega\theta}{a(1 - e^2)}$$

where

$$\omega = (1 - Z^2e^4/p_\theta^2c^2)^{1/2}$$

This is a precessing ellipse with a precession per orbit of

$$\delta\theta = 2\pi(1/\omega - 1)$$

## FLUID MECHANICS

Although matter seems to be composed of discrete particles, it is a useful approximation for many purposes to consider it to be a continuous fluid characterized by a mass density $\rho(\vec{x},t)$, a velocity $\vec{v}(\vec{x},t)$, a pressure tensor $P(\vec{x},t)$, an internal energy per unit mass $U(\vec{x},t)$ and perhaps some additional variables such as thermal and electrical conductivities. These macroscopic variables obey certain partial differential equations. For a fluid that is adequately described by $\rho$, $\vec{v}$, P and U, the nonrelativistic equations of fluid mechanics are

$$\rho_{,t} + (\rho v^j)_{,j} = 0 \qquad (84a)$$

$$(\rho v^i)_{,t} + (\rho v^i v^j)_{,j} + P^{ij}{}_{,j} = 0 \qquad (84b)$$

$$(\rho v^2/2 + \rho U)_{,t} + [(\rho v^2/2 + \rho U)v^j]_{,j} + (v_i P^{ij})_{,j} = 0 \qquad (84c)$$

When a comma followed by t or j appears as a subscript, it denotes a partial derivative with respect to t or $x^j$. The physical content is most easily seen by integrating these equations over a volume V and using the divergence theorem to convert the second and third terms to surface integrals. The time derivatives can be taken out of the integrals, and one obtains

$$d/dt \int_V d^3x\ \rho + \int_S \rho v^j da_j = 0 \qquad (85a)$$

$$d/dt \int_V d^3x\ \rho v^i + \int_S (\rho v^i) v^j da_j = -\int_S P^{ij} da_j \qquad (85b)$$

$$d/dt \int_V d^3 (\rho v^2/2 + \rho U) + \int_S (\rho v^2/2 + \rho U) v^j da_j = -\int_S v_i P^{ij} da_j \qquad (85c)$$

We identify $\rho\vec{v}$ as the momentum per unit volume and $(\rho v^2/2 + \rho U)$ as the energy per unit volume, consisting of kinetic energy and internal energy. The first terms in the above equations are the rates of change of the mass, momentum and energy in the volume V. The second terms are the rates at which mass, momentum and energy flow out of the surface S that bounds V. The right hand side of Eq.(85b) is the net force exerted by the surrounding fluid on the fluid within V; this force changes the momentum of the fluid within V. The right hand side of Eq.(85c) is the rate at which the surrounding fluid does work on the fluid within V, thereby changing its energy. We shall call Eqs.(85a,b,c) the conservation equations for mass, momentum and energy respectively.

We wish to find 4-dimensional tensor equations that reduce to Eqs.(85a,b,c) in the nonrelativistic limit. First, let us assume that the fluid is composed of particles of mass m with a particle density of $n(\vec{x},t)$ particles per unit volume. Consider a Lorentz transformation from the laboratory coordinate system to a coordinate system moving with an element of the fluid. We know that $\sqrt{(-g)}d^4x$ is an invariant volume element, and since $g = \eta = -1$

$$d^3x dt = d^3x' dt' \tag{86a}$$

where the primed variables refer to the coordinate system moving with the fluid. Now, $dt'$ is the differential of proper time, so $dt' = (1 - v^2/c^2)^{1/2} dt$, and

$$d^3x = (1 - v^2/c^2)^{1/2} d^3x' \tag{86b}$$

There must be the same number of particles in $d^3x'$ as there is in $d^3x$, so

$$n d^3x = n_0 d^3x' \tag{86c}$$

where

$$n_0 = n(1 - v^2/c^2)^{1/2} \tag{86d}$$

is the particle density measured in the frame of reference in which the particles are at rest. We call $n_0$ the <u>proper number density</u>. It is the number density that has an invariant significance. We rewrite the conservation of particles equation

$$n_{,t} + (nv^j)_{,j} = 0 \qquad (87a)$$

in terms of $n_0$ and multiply and divide the first term by c to obtain

$$\frac{\partial}{\partial ct}\left(\frac{n_0 c}{(1-v^2/c^2)^{1/2}}\right) + \frac{\partial}{\partial x^j}\left(\frac{n_0 v^j}{(1-v^2/c^2)^{1/2}}\right) = 0 \qquad (88)$$

Multiplying this by the mass of a particle m and defining

$$\rho_0 = mn_0 = \text{proper mass density} \qquad (89a)$$

$$u^\mu = \gamma(c, \vec{v}) = \text{fluid 4-velocity} \qquad (89b)$$

we obtain the relativistic form of Eq.(84a)

$$(\rho_0 u^\mu)_{,\mu} = 0 \qquad (89c)$$

This is the conservation equation for proper mass or the number of particles.

The conservation equations for momentum and energy, Eqs.(84b,c), constitute four first order partial differential equations. This suggest that that their relativistic generalization is

$$T^{\mu\nu}{}_{,\nu} = 0 \qquad (90)$$

where $T^{\mu\nu}$ is some second rank tensor constructed from $\rho_0$, $u^\mu$, $P^{\mu\nu}$ and U. In order to simplify the following discussion we shall assume that the pressure is a scalar, so that $P^{\mu\nu} = p\eta^{\mu\nu}$.
We expect that T is something like

$$T^{\mu\nu} = \rho u^\mu u^\nu - \eta^{\mu\nu} p \qquad (91)$$

However, because of the equivalence of mass and energy we expect the internal energy to contribute to the effective mass density of the fluid. A little experimentation shows that

$$T^{\mu\nu} = (\rho_0 + \rho_0 U_0/c^2 + P_0/c^2)u^\mu u^\nu - \eta^{\mu\nu}P_0 \qquad (92)$$

gives equations that reduce to Eqs.(84b,c) in the nonrelativistic limit, as will be shown. We have attached subscripts 0 to $P_0$ and $U_0$ to indicate that they are the pressure and internal energy per unit mass measured in a reference system in which they are at rest. We call $P_0$ the proper pressure and $U_0$ the proper internal energy per unit mass.

Letting $\mu = i = 1,2,3$ in Eq.(90), we obtain

$$\partial/\partial ct\, [(\rho_0 + \rho_0 U_0/c^2 + P_0/c^2)u^i u^0] \qquad (93)$$

$$+ \partial/\partial x^j\, [(\rho_0 + \rho_0 U_0/c^2 + P_0/c^2)u^i u^j + P_0\delta^{ij}] = 0$$

In the nonrelativistic limit $c \to \infty$, this reduces to Eq.(84b) since $u^i \to v^i$, $u^0 \to c$ and the terms with $c^2$ in the denominator drop out. Letting $\mu = 0$, we obtain

$$\partial/\partial ct\, [(\rho_0 + \rho_0 U_0/c^2 + P_0/c^2)u^0 u^0 - P_0]$$

$$+ \partial/\partial x^j\, [(\rho_0 + \rho_0 U_0/c^2 + P_0/c^2)u^0 u^j] = 0 \qquad (94a)$$

In the nonrelativistic limit, $v \ll c$ and

$$u^0 \approx c + v^2/2c, \quad u^i \approx v^i \qquad (94b)$$

In the first term of Eq.(94a), the terms containing $P_0$ cancel in this limit and we are left with

$$[(\rho_0 + \rho_0 U_0/c^2)u^0 u^\mu]_{,\mu} + [P_0 u^0 u^j/c^2]_{,j} = 0 \qquad (94c)$$

Now writing

$$(\rho_0 u^0 u^\mu)_{,\mu} = [\rho_0(c + v^2/2c)u^\mu]_{,\mu} = [\rho_0 v^2 u^\mu/2c]_{,\mu} \quad (94d)$$

where Eq.(89c) has been used. We are left with

$$[(\rho_0 v^2/2 + \rho_0 U_0)u^\mu]_{,\mu} + [P_0 v^j/c]_{,j} \quad (94e)$$

which becomes Eq.(84a) in the nonrelativistic limit.

To summarize: the nonrelativistic equations of fluid mechanics, Eqs.(84a,b,c) are replaced by the relativistic equations of conservation of mass energy and momentum, Eqs.(89c) and (90), with T given by Eq.(92) and called the <u>energy-momentum-tensor</u> of the fluid. The relativistic equations are easily generalized, so that they are valid in any coordinate system by simply replacing ordinary derivatives by covariant derivatives; thus

$$(\rho_0 u^\mu)_{;\mu} = 0 \quad (95a)$$

$$T^{\mu\nu}{}_{;\nu} = 0 \quad (95b)$$

The only change that need be made in T is the replacement of $\eta^{\mu\nu}$ by $g^{\mu\nu}$ in Eq.(92).

In the absence of dissipative effects, we expect entropy to be conserved. It is interesting to see how this emerges from the relativistic fluid equations. To this end we contract Eq.(90) with the 4-velocity to obtain a scalar that vanishes. thus

$$u_\mu \partial_\nu T^{\mu\nu} = u_\mu \partial_\nu [(\rho_0 + \rho_0 U_0/c^2 + P_0/c^2)u^\mu u^\nu - \eta^{\mu\nu} P_0] \quad (96a)$$

Using $u_\mu u^\mu = c^2$ and Eq.(89c), one may show that

$$u_\mu \partial_\nu u^\mu u^\nu = c^2 \partial_\nu u^\nu$$

$$= -(c^2/\rho)u^\nu \partial_\nu \rho = c^2 \rho u^\nu \partial_\nu (1/\rho) \quad (96b)$$

Then, a straightforward calculation leads to

$$u^{\nu}\partial_{\nu}U_0 + P_0 u^{\nu}\partial_{\nu}(1/\rho) \qquad (96c)$$

This may be written as

$$dU_0/d\tau + P_0 d/d\tau(1/\rho) \qquad (96d)$$

where

$$d/d\tau = dx^{\nu}/d\tau\ \partial/\partial x^{\nu} = u^{\nu}\partial_{\nu} \qquad (96e)$$

is the convective derivative operator. When it operates on a quantity, it gives the rate of change of that quantity as it moves with along a world line of an element of the fluid. From the second law of thermodynamics we identify

$$dU_0 + P_0 d(1/\rho) = T_0\ dS_0 \qquad (96f)$$

where $S_0$ is the proper entropy per unit mass, and $T_0$ is the proper absolute temperature. Our final result is

$$dS_0/d\tau = u^{\nu}\partial_{\nu}S_0 = 0 \qquad (96g)$$

The entropy, measured in a reference system moving with the fluid, remains constant.

We now turn our attention to the forces exerted on an electrically charged and conducting fluid. The Lorentz force per unit volume is

$$\vec{F} = \rho_e\vec{E} + 1/c\ \vec{J} \times \vec{B} \qquad (97)$$

We have attached a subscript e on the electric charge density $\rho_e$ to distinguish it from the mass density which we have denoted by $\rho$. One may easily check that the three components of $\vec{F}$ are the spatial components of

$$F^{\mu} = 1/c\ F^{\mu\nu}J_{\nu} = \text{4-force per unit volume} \qquad (98)$$

Lowering indices and substituting for $J^\nu$ from Eq.(12d) gives

$$F_\mu = (1/4\pi) F_{\mu\nu} F^{\nu\lambda}{}_{,\lambda}$$

$$= (1/4\pi)(\partial_\lambda [F_{\mu\nu}F^{\nu\lambda}] - F^{\nu\lambda}F_{\mu\nu,\lambda})$$

$$= (1/4\pi)(\partial_\lambda [F_{\mu\nu}F^{\nu\lambda}] - F^{\nu\lambda}\ 1/2\ (F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu}))$$

$$= (1/4\pi)(\partial_\lambda [F_{\mu\nu}F^{\nu\lambda}] + 1/2\ F^{\nu\lambda}F_{\nu\lambda,\mu})$$

$$= T_\mu{}^\lambda{}_{,\lambda} \qquad (99a)$$

where

$$T_\mu{}^\lambda = (1/4\pi)(F_{\mu\nu}F^{\nu\lambda} + 1/4\ (F^{\alpha\beta}F_{\alpha\beta})\delta_\mu^\lambda) \qquad (99b)$$

is called the <u>electromagnetic energy-momentum tensor</u> (or sometimes it is called the stress-energy-momentum tensor). In the above derivation we have used Eq.(14b).

Writing out the components of the energy-momentum tensor, one finds

$$T_\mu{}^\lambda = \begin{bmatrix} U & c\vec{G} \\ -c\vec{G} & P_i{}^j \end{bmatrix} \qquad (100a)$$

where

$$T_0{}^0 = U = (E^2 + B^2)/8\pi = \text{energy density} \qquad (100b)$$

$$T_i{}^0 = - T_0{}^i = cG^i = (\vec{E} \times \vec{B})^i/4\pi = c(\text{momentum density})^i \qquad (100c)$$

$$T_i{}^j = P_i{}^j = (E^iE^j + B^iB^j)/4\pi - \delta_i{}^j(E^2 + B^2)/8\pi$$

$$= \text{electromagnetic stress 3-tensor} \qquad (100d)$$

$\vec{G} = ?$

Calculating the force from the divergence of the energy-momentum tensor gives

$$F^i = -\partial G^i/\partial t + P^{ij}{}_{,j} \tag{101a}$$

for the spatial components $i = 1,2,3$. In addition to the divergence of the stress 3-tensor, there is a term representing the reaction to the change in the electromagnetic momentum, as was to be expected. For $\mu = 0$ we find

$$F^0 = 1/c\ (\partial U/\partial t + \nabla\cdot\vec{S}) \tag{101b}$$

where $\vec{S} = c^2\vec{G}$ is the Poynting vector. This term represents a rate of change of energy density due to an explicit time dependence (first term) and due to a flow of energy (second term).

We can now define a total energy-momentum tensor by adding the energy-momentum tensor for the electromagnetic field to that for the fluid to obtain

$$T^{\mu\nu} = T^{\mu\nu}_{(m)} + T^{\mu\nu}_{(e)} \tag{102a}$$

where the tensor to which we have attached the subscript (m) to denote matter is given by Eq.(92), and the tensor to which we have attached the subscript (e) to denote the electromagnetic field is given by Eq.(100). The conservation of energy and momentum for the combined system of interacting fluid and field is given by the very concise expression

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{102b}$$

We have written it in generally covariant form by replacing partial derivatives by covariant derivatives.

<u>Problem 7.</u>

From

$$u_\mu \partial_\nu (T^{\mu\nu}_{(m)} + T^{\mu\nu}_{(e)}) = 0$$

Show that the rate of change of proper entropy is given by

$$dS_0/d\tau = (1/\rho_0 T_0)\gamma[\vec{E} + (\vec{v} \times \vec{B})/c]\cdot[\vec{J} - \rho_e \vec{v}]$$

This has a simple interpretation. $\gamma[\vec{E} + (\vec{v} \times \vec{B}/c]$ is the electric field in a reference system moving with an element of fluid. $[\vec{J} - \rho_e \vec{v}]$ is the part of the current not due to convection. The scalar product of these two vectors gives the rate of production of ohmic heat. Dividing by $\rho_0 T_0$ gives the rate of entropy production per unit mass.

## INTERACTION OF A CHARGED PARTICLE WITH ITS OWN FIELD

Certain problems arise when one considers the interaction of a particle with its own electromagnetic field. If the particle is considered to be a point, then the field at its position is infinite. If the particle is considered to have finite dimensions, then one must explain why the particle is not blown apart by the disruptive force of its own field. These questions involve the structure of elementary particles and cannot be satisfactorily answered at this time, but we shall go into them in sufficient depth for some interesting results to emerge.

As a preliminary calculation, we shall use Eq.(52a) to calculate the electric field near a slowly moving particle of charge e and position $\vec{a}(t)$. We calculate the field at the time $t = 0$, and assume that at this time the particle is passing through the origin with zero velocity but with nonzero acceleration. This is not a very restrictive assumption, since we can always choose a Lorentz frame that moves with the velocity that the particle had at $t = 0$ and with its origin at the position of the particle at this time; Then we can make a Lorentz transformation to another coordinate system moving relative to the first and with its origin displaced. The position of the particle can be written as the Taylor series

$$\vec{a}(t) = \ddot{\vec{a}}t^2/2! + \dddot{\vec{a}}t^3/3! + \cdots \quad (103)$$

The dots over a symbol denote derivatives with respect to t, and these derivatives are to be evaluated at time t = 0. We assume that the particle is always moving so slowly that $|\vec{a}(t)| \ll ct$.
To evaluate the electric field at time t = 0, the right hand side of Eq.(52a) must be evaluated at the retarded time $t' = -R/c \approx r/c$. At this time $\vec{a}/r$, $\vec{\beta}$ and $\dot{\vec{\beta}}r/c$ are all small quantities. We rewrite Eq.(52a) keeping only terms that are linear in these small quantities and obtain

$$\begin{aligned}\vec{E}/e &= \vec{x}/r^3\,(1 + 3\vec{x}\cdot\vec{a}/r^2 + 3\vec{x}\cdot\vec{\beta}/r)\\ &\quad - \vec{a}/r^3 - \vec{\beta}/r^2 + (\hat{n}\hat{n}\cdot\dot{\vec{\beta}} - \dot{\vec{\beta}})\end{aligned} \quad (104)$$

Calculating $\vec{a}$, $\vec{\beta}$ and $\dot{\vec{\beta}}$ at the retarded time $t' = -r/c$ gives

$$\vec{a}(-r/c) = \ddot{\vec{a}}r^2/2c^2 - \dddot{\vec{a}}r^3/6c^3 \quad (105a)$$

$$\vec{\beta}(-r/c) = -\ddot{\vec{a}}r/c^2 + \dddot{\vec{a}}r^2/2c^2 \quad (105b)$$

$$\dot{\vec{\beta}}(-r/c) = \ddot{\vec{a}}/c - \dddot{\vec{a}}r/c^2 \quad (105c)$$

Substituting these into Eq.(104) gives

$$\vec{E}/e = \vec{x}/r^3 - (1/2rc^2)\ddot{\vec{a}}\cdot(1 + \vec{x}\vec{x}/r^2) + 2\dddot{\vec{a}}/3c^3 \quad (106)$$

The first term is the Coulomb field of the particle, and the other terms are corrections due to the motion of the particle. We have carried the calculation out only as far as terms in the third derivative of the particle's position.

Now let us imagine that an elementary particle is of finite size with a spherically symmetric distribution of charge with charge density $\rho(\vec{x}) = \rho(r)$. The net force that the particle exerts on itself is

$$\vec{F} = \int d^3x \; \rho(\vec{x})\vec{E}(\vec{x}) \qquad (107a)$$

We use Eq.(106) to calculate $\vec{E}(\vec{x})$, replacing e by $\rho(\vec{x}')d^3x'$ and $\vec{x}$ by $\vec{x} - \vec{x}'$ and integrating to obtain

$$\vec{E}(\vec{x}) = \int d^3x' \rho(\vec{x}')\left\{\frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} - \frac{\ddot{\vec{a}} \cdot}{2c^2|\vec{x} - \vec{x}'|}\left[1 + \frac{(\vec{x} - \vec{x}')(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^2}\right] + 2\dddot{\vec{a}}/3c^3\right\} \qquad (107b)$$

The Coulomb force exerts no net force and we are left with

$$\vec{F} = (\delta m)\cdot\ddot{\vec{a}} + (2e^2/3c^2)\dddot{\vec{a}} \qquad (107c)$$

where we have used

$$e = \int d^3x \; \rho(\vec{x}) \qquad (107d)$$

and defined the electromagnetic mass tensor

$$(\delta m) = (1/2c^2)\int d^3x \int d^3x' \rho(\vec{x})\rho(\vec{x}') \; [1 + \vec{n}\vec{n}]|\vec{x} - \vec{x}'|^{-1} \qquad (107e)$$

where $\vec{n}$ is a unit vector pointing from $\vec{x}$ to $\vec{x}'$. One may easily show that the components of the electromagnetic mass tensor are

$$(\delta m)_{ij} = \delta_{ij}(4U/3c^2) \qquad (107f)$$

where

$$U = 1/2 \int d^3x \int d^3x' \; \rho(\vec{x}) \; \rho(\vec{x}') \; |\vec{x} - \vec{x}'|^{-1} \qquad (107g)$$

is the electrostatic energy of the charge distribution. Our final result for the self force, to which we now attach a subscript s, is

$$\vec{F}_s = -(4U/3c^2)\ddot{\vec{a}} + (2e^2/3c^3)\dddot{\vec{a}} \qquad (108)$$

We may add this force to the external force that acts on a particle and write the equations of motion as

$$(m + 4U/3c^2)\ddot{\vec{a}} - (2e^2/3c^3)\dddot{\vec{a}} = \vec{F}_{ext} \qquad (109)$$

Problem 8.

Show that

$$U = \int d^3x \; |\vec{E}(\vec{x})|^2/8\pi$$

Assume that the charge density of a particle is constant inside a sphere of radius $r_0$ and zero outside. Show that

$$U = 3e^2/5r_0$$

If one attempts to account for all of a particles mass as electromagnetic mass, then one must conclude that the particle's radius is of order $e^2/mc^2$, the precise numerical factor depending on how the charge is distributed. This is called the electromagnetic radius and is of order $10^{-13}$cm. for the electron.

If we had kept terms beyond the third derivative in Eq.(103) then there would have been additional terms in the equations of motion. Let us now suppose that the radius of the particle is reduced to zero. We would find that the coefficient of the fourth derivative of $\vec{a}$ is proportional to the radius, the coefficient of the fifth derivative is proportional to the radius squared and so on. These higher derivatives vanish from the equations of motion in the limit that the particle becomes a point. Only the coefficient of the third derivative, $(2e^2/3c^3)$, is independent of the radius. As Problem 8 showed, the electromagnetic mass diverges inversely as the radius, which may seem to preclude the possibility of point particles. However, note that the electromagnetic mass appears together with the ordinary mass m. There is no way to separate the two. We cannot turn off the electric charge of the particle and measure m. The mass that we measure experimentally is

$$m_{exp} = m + 4U/3c^3 \tag{110}$$

and we know that this is finite. We should write the equations of motion of a point particle as

$$m_{exp}\ddot{\vec{a}} - (2e^2/3c^3)\,\dddot{\vec{a}} = \vec{F}_{ext} \tag{111}$$

This process of absorbing infinite contributions to the mass into the finite experimental mass is known as <u>mass renormalization</u>. Renormalization is an essential feature of modern quantum field theory. We have seen an elementary example of renormalization here.

The term in the self force

$$\vec{F}_{rad} = (2e^2/3c^3)\,\dddot{\vec{a}} \tag{112}$$

is called the <u>radiative reaction force</u>. In order to gain some insight into the physical origin of this force, let us consider the work done by this force during a time interval from $t_1$ to $t_2$. We shall suppose that the acceleration of the particle vanishes at the beginning and end of this interval. The work is

$$\begin{aligned} W = \int dt\ \vec{v}\cdot\vec{F}_{rad} &= (2e^2/3c^3)\int dt\ \dot{\vec{a}}\cdot\dddot{\vec{a}} \\ &= -(2e^2/3c^3)\int dt\ \ddot{\vec{a}}\cdot\ddot{\vec{a}} \end{aligned} \tag{113}$$

where we have integrated by parts in the last step. This is just the energy that we would calculate that was radiated by using the Larmor formula, Eq.(54). As the particle loses energy by radiation there must be reaction on the motion of the particle.

In what follows we shall drop the subscripts exp on m and ex on $\vec{F}$ and understand that m is the experimental mass and $\vec{F}$ is the external force. We write the equations of motion as

$$m(\ddot{\vec{a}} - \tau\dddot{\vec{a}}) = \vec{F} \tag{114a}$$

where

$$\tau = 2e^2/3mc^3 \approx 10^{-23}\text{sec.} \qquad (114b)$$

This can be written as

$$d/dt\ (\dddot{\vec{a}}\ e^{-t/\tau}) = -\ (1/m\tau)e^{-t/\tau}\vec{F}(t) \qquad (114c)$$

Now, suppose the force is zero. There are two solutions:

$$\ddot{\vec{a}} = 0, \quad \text{and } \ddot{\vec{a}} = Ce^{+t/\tau} \qquad (114d)$$

The first of these is the solution we expect from Newtonian mechanics. The second solution with an exponentially growing acceleration is clearly unphysical. By replacing the second order differential equation of Newtonian mechanics by a third order equation, we have introduced this additional and unwanted solution. We may abolish this unwanted solution and make the equations of motion second order again by integrating both sides of Eq.(114c) from t to ∞ and assuming that the acceleration is always finite. We obtain

$$m\ddot{\vec{a}} = 1/\tau \int_t^\infty dt' e^{(t-t')/\tau}\ \vec{F}(t')$$

$$= 1/\tau \int_0^\infty ds e^{-s/\tau}\ \vec{F}(t + s) \qquad (115)$$

In the second step we have changed the variable of integration to $s = t' - t$.

Eq.(115) is an integrodifferential equation. Since only the second derivatives with respect to time occur, the motion is determined by the position and velocity at any instant for a given force. However, the motion of the particle at time t depends on forces at times greater than t. We are accustomed to thinking of the force as the cause and the motion as the effect; we now find that the effect precedes the cause, although only by times of the order of $10^{-23}$ for an electron. Our investigations have led us into an area in which quantum effects are important and classical mechanics is not valid.

Problem 9.

Consider the motion of a particle in response to an impulsive force $\vec{F}(t) = \delta(t)$. Show that the particles velocity is

$$\vec{v}(t) = \begin{cases} \vec{v}(-\infty) + (1/m)e^{t/\tau} & \text{for } t < 0 \\ \vec{v}(-\infty) + (1/m) & \text{for } t > 0 \end{cases}$$

The radiative reaction force, Eq.(112), is clearly a nonrelativistic expression. In our derivation we assumed that the velocity was small. We would like to generalize this formula to arbitrary velocities. $\vec{F}_{rad}$ must be the nonrelativistic approximation to the spatial components of some 4-vector $F^{\mu}_{rad}$. We can add to Eq.(112) a term proportional to the velocity, since that term would vanish in the limit of small velocities. We are led to write the radiative reaction 4-vector force as

$$F^{\mu}_{rad} = 2e^2/3c^3\, [\dddot{a}^{\mu} + \alpha\dot{a}^{\mu}] \qquad (116a)$$

where $\alpha$ is some scalar still to be determined. In order for this to be a 4-vector we must interpret the dots over symbols to denote derivatives with respect to proper time. Taking the scalar product of the equations of motion with the 4-velocity gives

$$m(\dot{a}_{\mu}\ddot{a}^{\mu}) = md/da\,(\dot{a}_{\mu}\dot{a}^{\mu}/2) = md/da(c^2/2) = 0 = \dot{a}_{\mu}F^{\mu}_{rad}$$

$$= (2e^2/3c^3)[\dot{a}_{\mu}\dddot{a}^{\mu} + \alpha c^2] \qquad (116b)$$

from which

$$\alpha = -\dot{a}_{\mu}\dddot{a}^{\mu}/c^2 = \ddot{a}_{\mu}\ddot{a}^{\mu} \qquad (116c)$$

Our final expression for the radiative reaction 4-vector force is

$$F^{\mu}_{rad} = 2e^2/3c^3\, [\dddot{a}^{\mu} + \dot{a}^{\mu}(\ddot{a}_{\nu}\ddot{a}^{\nu})/c^2]$$

(116d)

UNITS AND DIMENSIONS

The subject of units and dimensions is often a confusing one. The student beginning his study of physics with mechanics learns that physical quantities have the dimensions of length, mass and time. These may be measured in various systems of units such as feet or meters for length, seconds or days for time, slugs or grams for ~~length~~ MASS and so forth. When electromagnetic theory is studied, electric charge may or may not be introduced as a separate dimension depending on whether the S.I. or the Gaussian system of units is used. When thermodynamics is studied, temperature may be introduced as a separate dimension. It may be noted that with the addition of each new physical dimension, a new fundamental constant is added to the equations of physics. For instance, in the S.I. system where charge (or equivalently, current) is a dimension, the permittivity $\epsilon_0$ and permeability $\mu_0$ of the vacuum appear in Maxwell's equations, but in the Gaussian system only one constant is needed, the velocity of light c. Along with the introduction of temperature, one adds Boltzmann's constant k to the table of fundamental constants.

With the increase in our knowledge of physics, it has become possible to eliminate some dimensions and their accompanying fundamental constants. When kinetic theory and statistical mechanics were developed in the last century, it was found that in thermal equilibrium at temperature T, each degree of freedom of a mechanical system had an energy of kT/2. Therefore, temperature and energy are equivalent and Boltzmann's constant $k = 1.38 \times 10^{-16}$ ergs per degree Kelvin is the conversion factor between the two. If some clever experimentalist had designed a thermometer that measured temperature directly in energy units at the time when the theory of heat was being developed, then the Fahrenheit, Celsius and Kelvin scales of temperature need never have been introduced and Boltzmann's constant would be absent from tables of physical constants and from the equations of physics. There may be some advantage for experimental physicists and engineers to write their equations in the units that they measure with mercury thermometers, but for the discussion of fundamental physical laws it is an unnecessary encumbrance.

When Einstein's special theory of relativity was published in 1905, it became apparent that we inhabit a four dimensional universe with time as the fourth dimension. We conventionally measure distances along the time axis in different units (seconds, for instance) than are used for distances along the three spatial axes (centimeters, for instance), and the conversion factor between these different units is the velocity of light $c = 3 \times 10^{10}$ cm./sec. Our practice in this respect is like that of sailors, who measure horizontal distances in nautical miles and vertical distances in fathoms. To the ordinary sailor there must seem to be a great difference between travel in the horizontal and vertical directions, but a sailor with a philosophical mind would realize that horizontal and vertical distances are fundamentally the same and can be measured in the same units. We can imagine building a clock that measures time directly in units of length. For instance, we might construct a laser with a Kerr cell shutter that emits a brief pulse of light that travels a known distance, say half a kilometer, to a mirror where it is reflected back to the source. Its return is recorded and it triggers another pulse that follows the same path. The pulse of light shuttling back and forth between laser and mirror plays the role of the pendulum in a mechanical clock. Each round trip is recorded as the passage of one kilometer of time. With time measured in this way, seconds need never have been introduced and there is no need for a conversion factor. The velocity of light drops out of all of the equations of physics; it is just the dimensionless number 1 cm. of length per cm. of time. Time has been abolished as a separate physical dimension, and with its disappearance a conversion factor vanishes.

An alternative to the above scheme for measuring time that is closer to what is done in practice is the following: One chooses as a standard of length the wavelength of a certain spectral line, and one choses as a standard of time the period of the same spectral line. Then the velocity of light is clearly unity. This is close to our present choice of standards, but different spectral lines are chosen for the length and time standards. The standard of length is the wavelength of the $2p_{10}$ to $5d_5$ transition in the $^{86}Kr$ atom, and the standard of time is the period of a transition between two hyperfine levels of the ground state of th $^{133}Cs$ atom.

With the above schemes for measuring time directly in units of length, the question of the variation of the velocity of light, as has sometimes been suggested, cannot arise. It is as nonsensical as asking if the number of centimeters per inch varied with time. The velocity of light is constant and unity by definition. The velocity of light could vary only if we had separate definitions for the standards of length and time. For instance, we could choose the standard of length to be the length of a meter bar and the standard of time to be the period of the earths rotation on its axis, the day. With this choice of standards, the velocity of light can, and in fact does, change with time. However, it seems preferable to take the point of view that the velocity of light is constant and that the period of the earth's rotation decreases with time due to tidal friction.

Relativity has allowed us to abolish time as a physical dimension. Quantum mechanics allows us to abolish mass as a physical dimension. DeBroglie's relation

$$\vec{p} = \hbar \vec{k} \tag{117a}$$

associates a wavelength $\lambda = 2\pi/|\vec{k}| = 2\pi\hbar/|\vec{p}|$ with the momentum $\vec{p}$. In relativistic quantum mechanics this is generalized to

$$p^{\mu} = \hbar k^{\mu} \tag{117b}$$

relating the 4-vector momentum $p^{\mu}$ to the wave 4-vector. This relation allows us to assign the physical dimension of inverse length to every momentum with $\hbar$ as the conversion factor. For a particle at rest, only $p^0 = mc$ is nonvanishing, and the length associated with the rest mass m is

$$\bar{\lambda} = \lambda/2\pi = \hbar/mc \tag{117c}$$

This is the Compton wavelength of the particle. The reciprocal of this wavelength, measured in cm.$^{-1}$ say, describes the mass of the particle just as well as the mass measured in grams. We may regard $\hbar/c = 3.52 \times 10^{-38}$ gm.-cm. as the conversion factor between mass measured in grams and mass measured in cm.$^{-1}$. If we agree to measure mass in units of inverse length, then mass as a separate physical dimension and the accompanying conversion factor $\hbar/c$ can be abolished from physics.

Let us agree to measure time in units of length and mass in units of inverse length. We shall also measure temperature in energy units, but since energy now has dimensions of inverse length, so does temperature. This set of units is called <u>natural units</u>. There is only one physical dimension, length, and all physical quantities have dimensions of some power of length, denoted by $L^n$. We have already seen that in this set of units length and time have dimensions L while mass, momentum, energy and temperature have dimensions $L^{-1}$. It follows that velocity is dimensionless, ($L^0$), and acceleration has dimensions $L^{-1}$. From Newton's second law we see that force must have dimensions $L^{-2}$. Then, from Coulomb's law we see that electric charge is dimensionless, and then it follows from the formula for the Lorentz force that the electric and magnetic fields have dimensions $L^{-2}$. Since these fields are given by derivatives of potentials, it follows that the potentials have dimensions $L^{-1}$. Finally, from Newton's law of gravitation it follows that the gravitational constant G has dimensions $L^2$. These dimensions of physical quantities are summarized in Table 1. We also give the dimensions in a system with length (L), time (T), mass (M) and degrees Kelvin (K) as fundamental dimensions. Useful numerical values are given in Table 2. In natural units the physical constants c, $\hbar$ and k are unity by definition and are dimensionless; they never appear in any of the equations of physics. The question of their possible variation can never arise.

Table 1.

| Quantity | LTMK Dimension | Natural Dimension |
| --- | --- | --- |
| length | $L$ | $L$ |
| time | $T$ | $L$ |
| mass | $M$ | $L^{-1}$ |
| temperature | $K$ | $L^{-1}$ |
| velocity | $LT^{-1}$ | $L^{0}$ |
| acceleration | $LT^{-2}$ | $L^{-1}$ |
| momentum | $MLT^{-1}$ | $L^{-1}$ |
| energy | $ML^{2}T^{-2}$ | $L^{-1}$ |
| force | $MLT^{-2}$ | $L^{-2}$ |
| $\vec{E}$, $\vec{B}$ | $M^{1/2}L^{-1/2}T^{-1}$ | $L^{-2}$ |
| $\phi$, $\vec{A}$ | $M^{1/2}L^{1/2}T^{-1}$ | $L^{-1}$ |
| charge | $M^{1/2}L^{3/2}T^{-1}$ | $L^{0}$ |
| entropy | $ML^{2}T^{-2}K^{-1}$ | $L^{0}$ |
| gravitational const., G | $M^{-1}L^{3}T^{-2}$ | $L^{2}$ |
| Boltzmann's const., k | $ML^{2}T^{-2}K^{-1}$ | $L^{0}$ |
| Planck's const., $\hbar$ | $ML^{2}T^{-1}$ | $L^{0}$ |

Table 2.

| Quantity | LTMK | Natural |
| --- | --- | --- |
| e = quantum of charge | $4.8 \times 10^{-10}$ gm.$^{1/2}$cm.$^{3/2}$sec.$^{-1}$ | $0.085 = (137)^{-1/2}$ |
| c = velocity of light | $3 \times 10^{10}$ cm./sec. | 1 |
| $\hbar$ = Planck's const. | $1.05 \times 10^{-27}$ erg. sec. | 1 |
| $\hbar/c$ | $3.52 \times 10^{-38}$ gm. cm. | 1 |

| | | |
|---|---|---|
| m = electron mass | $9.1 \times 10^{-28}$ gm. | $2.6 \times 10^{10}$ cm.$^{-1}$ |
| G = grav. const. | $6.67 \times 10^{-8}$ cm.$^3$gm.$^{-1}$sec.$^{-2}$ | $2.6 \times 10^{-66}$ cm.$^2$ |

When natural units are used, dimensional analysis is very simple since there is only one dimension, length. As the following problems show, it is often easy to guess the answer to questions to within a numerical factor. After an answer has been guessed in natural units, if one wants the answer in conventional units, one merely supplies sufficient factors of c and $\hbar/c$ to restore the dimensions of time and mass. In the problems that follow we give the answers in natural units obtained by dimensional analysis and then give the answers in conventional units with the correct numerical factors when these are known.

Problem 10.

In the preceding section we calculated the force that a charged particle exerts on itself in the form

$$\vec{F}_s = \sum_n C_n \, d^n\vec{a}/dt^n$$

where $C_n$ must depend on the radius $r_0$ of the particle. We omitted terms with $n > 3$, and claimed that they vanished in the limit of a point particle. Use dimensional analysis to find the dependence of $C_n$ on $r_0$. Show that

$$C_n \approx e^2 \, r_0^{n-3}$$

$$= k_n e^2 r_0^{n-3}/c^2$$

where $k_n$ is a numerical factor.

Problem 11.

When a star without angular momentum collapses to a black hole, the radius of the black hole can only depend on the mass of the star (and fundamental constants, of course). Show that this radius is

$$R_B \approx GM$$
$$= 2GM/c^2$$

A sphere of this radius is called the <u>horizon</u> of the black hole. Particles and photons can fall into the black hole through the horizon, but nothing can ever pass outward through the horizon.

<u>Problem 12.</u>

(a) Show that the energy per unit volume u inside a hohlraum (a furnace with walls maintained at a temperature T) is

$$u \approx T^4$$

$$= (\pi^2/15)(kT)^4/\hbar^3c^3$$

(b) Show that the emmitance E (energy emitted per unit area per unit time) from a black body at temperature T is

$$E \approx T^4$$

$$= (\pi^2/60)(kT)^4/\hbar^3c^2$$

<u>Problem 13.</u>

A theorem due to S. W. Hawking states that the surface area of the horizon of a black hole cannot decrease and generally increases in a dynamical process. This irreversibility in the growth of black hole surface area suggested to Jacob Beckenstein an analogy with the second law of thermodynamics and led him to propose that a black hole be assigned an entropy that is proportional to its area. Show that the entropy of a black hole of mass M must be

$$S \approx GM^2$$

$= 4\pi kGM^2/\hbar c$

Now, use the thermodynamic identity TdS = dU + PdV to show that a black hole must have a temperature of

$$T = \hbar c^3/8\pi kGM$$

A black hole must lose energy by radiation, and as it does so its mass must decrease. Show that a black hole of mass M must evaporate by radiation in a time

$$t = 5120\pi M^3 G^2 \text{ (natural units)}$$

Show that for a black hole to survive from the time of the big bang (about $10^{10}$ years ago) untill the present, its initial mass must exceed $10^{14}$ grams = $10^8$ tons. This is about the mass of a small asteroid.

Another set of units that is useful in calculations when gravitational effects are important but quantum effects are of lesser importance are the geometrized units. In these units c = 3 x $10^{10}$ cm./sec. is used as the conversion factor between time and length as before, and $G/c^2 = 0.74 \times 10^{-28}$ cm./gm. is used as a conversion factor between mass and length. The length associated with a mass is the radius of the black hole of this mass. As in natural units, temperature is measured in energy units. In geometrized units, c = k = G = 1, whereas in natural units c = k = $\hbar$ = 1. In Table 3 we give the dimensions of important physical quantities in geometrized units.

Table 3.

| Quantity | Geometrized dimension |
|---|---|
| length | L |

| | |
|---|---|
| time | $L$ |
| mass | $L$ |
| temperature | $L$ |
| velocity | $L^0$ |
| acceleration | $L^{-1}$ |
| momentum | $L$ |
| energy | $L$ |
| force | $L^0$ |
| E, B | $L^{-1}$ |
| $\phi$, $\vec{A}$ | $L^0$ |
| charge | $L$ |
| entropy | $L^0$ |
| gravitational const., G | $L^0$ |
| Boltzmann's const., k | $L^0$ |
| Planck's const. h | $L^2$ |

The square root of the gravitational constant in natural units is a length called the Planck length with the value

$$l_p = \sqrt{G} = 1.6 \times 10^{-33} \text{ cm}.$$

If one chooses this as the unit of length, one obtains a set of units called Planck units in which all quantities are dimensionless and $c = k = \hbar = G = 1$.

# CHAPTER 9

## ACTION-AT-A-DISTANCE ELECTRODYNAMICS AND THEORIES OF GRAVITATION.

### WHEELER-FEYNMAN ELECTRODYNAMICS.

We shall now discuss a theory of classical electrodynamics whose modern formulation is due to Wheeler and Feynmann. In this formulation the electromagnetic field can be eliminated from the theory, and the particles interact directly with one another.

Our starting point is the action function for charged particles interacting with an electromagnetic field.

$$S = -\sum_a m_a c^2 \int da \;-\; \sum_a (e_a/c) \int A_\mu \dot{a}^\mu \, da$$
$$- (1/16\pi c) \int d^4x \; (-g)^{1/2} F_{\mu\nu} F^{\mu\nu}$$
$$= S_p + S_{pf} + S_f \qquad (1)$$

The notation is that of Eq.(79) of Chapter 8. As was shown there the requirement that S be an extremum, $\delta S = 0$, leads to the equations of motion for the particles and Maxwell's equations for the electromagnetic field.

Next, we write

$$A^\mu = \sum_a A_a^\mu + A^\mu{}_f \qquad (2)$$

where $A_a^\mu$ is the field produced by the particle labeled a given by Eq.(46) of Chapter 8, and $A_f^\mu$ is the free field (that is, the solution of $\Box^2 A_f^\mu = 0$). In a similar way we write $F_{\mu\nu}$ as a sum of fields produced by the particles and free fields, attaching indices a,b,c,---- and f where ever there is space for an index. We can now write the action as

$$S = -\sum_a m_a c^2 \int da$$
$$-\sum_a \sum_b (1/c^2) \int d^4x\, A^b_\mu\, J^\mu_a$$
$$-(1/16\pi c) \sum_a \sum_b \int d^4x\, (-g)^{1/2}\, F_{\mu\nu}{}^a F^{\mu\nu}{}_b$$
$$+ \text{ terms containing the free fields.} \qquad (3)$$

In the second term we have used Eq.(45) of Chapter 8 for the current due to particle a. We may use Maxwell's equations to write

$$J^\mu_a = (c/4\pi)(-g)^{-1/2}\ \partial_\nu[(-g)^{1/2}F^{\mu\nu}{}_a] \qquad (4)$$

We may substitute this into the second term and integrate by parts to show that

$$\text{2nd term} = (1/8\pi c)\sum_a \sum_b \int d^4x(-g)^{1/2}F^{\mu\nu}{}_a F_{\mu\nu}{}^b \qquad (5)$$

We can now combine the second and third terms and write the action as

$$S = -\sum_a m_a c^2 \int da$$
$$+ (1/16\pi c) \sum_a \sum_b \int d^4x(-g)^{1/2}F^{\mu\nu}{}_a\, F_{\mu\nu}{}^b$$
$$+ \text{ terms containing the free fields.} \qquad (6)$$

We shall take the terms with a = b in the second term and combine with the first term using

$$d^4x = c\ da\ d^3x\ dt/da \qquad (7)$$

to obtain

$$S = - \sum_a \int (m_a + \delta m_a)\, da$$
$$+ (1/16\pi c) \sum_{a \neq b}\sum \int d^4x (-g)^{1/2} F_{\mu\nu}^{\ a} F^{\mu\nu}_{\ \ b}$$
$$+ \text{terms containing the free fields.} \qquad (8a)$$

where

$$\delta m_a = - (1/16\pi c) \int d^3 (-g)^{1/2} (dt/da) F_{\mu\nu}^{\ a} F^{\mu\nu}_{\ \ a}$$
$$= (1/8\pi c^2) \int d^3x\ (-g)^{1/2} (dt/da)(E_a^2 - B_a^2) \qquad (8b)$$

We shall interpret this as the electromagnetic mass of particle a. Comparing it with Eq.(107) and Problem 8 of Chapter 8 we see that the expression given there is the nonrelativistic limit of the present expression except for a factor of 4/3. The integrand of Eq.(8b) is a scalar density as it should be. The factor of 4/3 is an artifact of the nonrelativistic calculation of the preceeding chapter. We shall renormalize the mass by writing $m_{aexp} = m_a + \delta m_a$ and identifying this as the experimental mass. We argue as we did in Chapter 8 that it is only the experimental mass that is observable, and we observe it to be finite whatever the two parts $m_a$ and $\delta m_a$ may be. We drop the subscript exp in all that follows.

Next we reverse the calculations that led from the second term of Eq.(1) to the second term of Eq.(3) to Eq.(5) to obtain

$$(1/8\pi c) \int d^4x (-g)^{1/2} F^{\mu\nu}_{\ \ a} F_{\mu\nu}^{\ \ b} = (e_a/c) \int A_\mu^{\ b}\, \dot{a}^\mu\, da \quad (9)$$

Next, we use Eq.(46) of Chapter 8 for $A_a^\mu$. Finally, we assume that there no free fields in the universe, so we discard the free field terms in Eq.(8a) and write our final expression for the action as

$$S = - \sum_a m_a c^2 \int da + \sum_{a \neq b}\sum (e_a e_b/c^2) \int da \int db\ \dot{b}_\mu \dot{a}^\mu\ \delta(ab \cdot ab) \quad (10)$$

This action functional is known as the Fokker-Tetrode action. The electromagnetic field has been completely eliminated. Charged particles interact when the square of their space-time separation ab·ab vanishes. The functional derivative of S with respect to each of the particles coordinates is set equal to zero to obtain the equations of motion of the particles. To see how this comes about let collect all of the terms in S that contain the coordinates and velocities of particle a and denote the sum of these terms by S(a). Thus

$$S(a) = - m_a c^2 \int da + \sum_{b \neq a} (e_a e_b / c) \int da \int db \, \dot{a}^\mu \dot{b}_\mu \delta(ab \cdot ab)$$

$$= - m_a c^2 \int da + (e_a / c) \int da \, \dot{a}^\mu A_\mu (a^\nu) \qquad (11a)$$

where

$$A_\mu (x^\nu) = \sum_{b \neq a} e_b \int db \, \dot{b}_\mu \delta(xb \cdot xb) \qquad (11b)$$

We recognize Eq.(11a) as the action for a particle in an electromagnetic field. Requiring $\delta S(a) = 0$ gives the particle equations of motion, Eq.(76b) of Chapter 8. We recognize Eq.(11b) as the electromagnetic potential due to all of the particles except particle a. We have reintroduced the electromagnetic field, but this time only as a mathematical fiction, in the process of demonstrating that the equations of motion followed from the action principle.

There seems to be one obvious flaw in this theory of electrodynamics; according to Eq.(11b) the interaction between particles propagates along the future light cone as well as along the past. This brings us face to face with a perenially puzzling question. Why is it, that in nature we only observe the retarded solutions of maxwells equations and never the advanced solutions? Maxwell's equations are invariant under the time reversal transformation:

$$t \to - t, \quad \vec{E} \to + \vec{E}, \quad \vec{B} \to - \vec{B}$$

$$\rho \to + \rho, \quad \vec{J} \to - \vec{J} \qquad (12)$$

What breaks this invariance?

The answer given by Wheeler and Feynmann is the following: If we lived in a universe that contained only a few particles, we would see both the retarded and advanced fields of a charged particle. However, we inhabit a universe containing an immense number of charged particles whose presence we cannot neglect. The behavior of one particle induces a responce in the rest of the universe that reacts back on the first particle. We shall treat this many particle system in the following approximate way. As in Eq.(34) of Chapter 8, we write the relation between the Fourier transforms of the current and the potential as

$$A^{\mu}(k) = \Delta^{(\mu)}(k)\, J^{\mu}(k) \qquad (13a)$$

where

$$\Delta(k) = -4\pi/ck^2 = -4\pi c/(\omega^2 - c^2|\vec{k}^2|) \qquad (13b)$$

We shall divide the current into two parts; a part due to a test particle that we imagine to be under our control, and an induced current in the rest of the universe. We shall write the induced current as the product of a conductivity $\sigma$ and the electric field. Thus

$$J^{\mu}(k) = J^{\mu}_{t}(k) + J^{\mu}_{i}(k) \qquad (14a)$$

$$\vec{J}_i(k) = \sigma(k)\vec{E}(k) = \sigma[(i\omega/c)\vec{A} - i\vec{k}\Phi] \qquad (14b)$$

$$J^{0}(k) = c\rho(k) = c\vec{k}\cdot\vec{J}/\omega \qquad (14c)$$

We substitute Eq.(14) into Eq.(13) and solve for the potential in terms of the current of the test particle to obtain

$$\vec{A} = \Delta_1\vec{J}_t + (\vec{\Delta}_2{}' - \vec{\Delta}_1{}')J^{0}_{t} \qquad (15a)$$

$$\phi = \Delta_2 J_t^0 \qquad (15b)$$

where

$$\Delta_1 = \frac{\Delta}{1 - i\omega\sigma\Delta/c} \qquad (15c)$$

$$\Delta_2 = \frac{\Delta}{1 - (i\omega\sigma\Delta/c)(1 - c^2K^2/\omega^2)} \qquad (15d)$$

$$\vec{\Delta}_1{}' = \omega\vec{K}\Delta_1/cK^2 \qquad (15e)$$

$$\vec{\Delta}_2{}' = \omega\vec{K}\Delta_2/cK^2 \qquad (15f)$$

where $K = |\vec{K}|$.

The terms that contain $\sigma$ are the corrections to the potential that result from taking into account the currents that are induced in the universe. To procede we must make some assumption about the conductivity. We shall adopt a simple model in which each particle in the universe moves in response to the electric field and is acted on by a damping force proportional to the velocity. We write the equations of motion as

$$d\vec{v}/dt + \nu\vec{v} = (e/m)\vec{E} \qquad (16a)$$

After fourier transformation, we solve to obtain

$$\vec{v} = ie\vec{E}/m(\omega + i\nu) \qquad (16b)$$

The current that flows in response to the electric field is

$$\vec{J} = ne\vec{v} = \sigma\vec{E} \qquad (16c)$$

where the conductivity is

$$\sigma = \frac{i\omega_p^2}{4\pi(\omega + i\nu)} \tag{16d}$$

In the above n is the particle density and $\omega_p = (4\pi ne^2/m)^{1/2}$ is the plasma frequency. It is not difficult to extend the model to a plasma of several species of particles, but nothin very interesting results.

Using this conductivity in Eqs.(15c,d) gives

$$\Delta_1 = \frac{-4\pi c(\omega + i\nu)}{(\omega^2 - \omega_K^2)(\omega + i\nu) + i\nu\omega_2^2} \tag{17a}$$

$$\Delta_2 = \frac{-4\pi c\omega(\omega + i\nu)}{(\omega^2 - c^2K^2)(\omega^2 + i\nu\omega - \omega_p^2)} \tag{17b}$$

where

$$\omega_K^2 = \omega_p^2 + c^2K^2 \tag{17c}$$

The poles of $\Delta_1$ are at the approximate values

$$\begin{aligned} \omega_1 &= \omega_K - i\nu\omega_p^2/2\omega_K^3 \\ \omega_2 &= -\omega_K - i\nu\omega_p^2/2\omega_K^3 \\ \omega_3 &= -i\nu \end{aligned} \tag{17d}$$

where we have treated $\nu$ and $\omega_p$ as small. All three of these poles lie in the lower half $\omega$-plane, so when we invert the Fourier transform, we find no response for $t < 0$ associated with $\Delta_1$. However, $\Delta_2$ has poles on the real $\omega$ axis at $\omega = \pm cK$, so these would give a response for both $t < 0$ and $t > 0$. If we calculate the electric field, we find

$$\vec{E} = (i\omega/c)\vec{A} - i\vec{K}\phi \qquad (18)$$

$$= (i\omega/c)\Delta_1\vec{J}_t - (i\omega^2/c^2K^2)\vec{K}\Delta_1 J_t^{\,0}$$

$$+ (i\vec{K}/c^2K^2)(\omega^2 - c^2K^2)\Delta_2 J_t^{\,0}$$

The factor $(\omega^2 - c^2K^2)$ that multiplies $\Delta_2$ cancels the poles on the axis, thus removing the response for $t < 0$.

It is possible and convenient to make a gauge transformation on the potential that eliminates the unwanted response for $t < 0$, since it is not present when the electromagnetic field is calculated. We write

$$\vec{A}' = \vec{A} + i\vec{K}\chi \qquad (19a)$$

$$\phi' = \phi + (i\omega/c)\chi \qquad (19b)$$

where

$$\chi = -(i\omega/cK^2)(\Delta_1 - \Delta_2)J_t^{\,0} \qquad (19c)$$

has been chosen to eliminate the unwanted terms. We find

$$\vec{A}' = \Delta_1\vec{J}_t \qquad (20a)$$

$$\phi' = (\Delta_3 - \Delta_4)J_t^{\,0} \qquad (20b)$$

where

$$\Delta_3 = (\omega^2/c^2K^2)\Delta_1 \qquad (20c)$$

$$\Delta_4 = (\omega^2/c^2K^2 - 1)\Delta_2 \qquad (20d)$$

In inverting the inverse Fourier transform, we shall only be interested in the limit of vanishing $\omega_p$,
since we are not interested in real plasma effects. In this limit

the results are very simple:

$$\Delta_1 = \Delta_3 = \Delta_r = (1/rc)\delta(r - ct) \qquad (21a)$$

$$\Delta_4 = 0 \qquad (21b)$$

$$A^\mu(x) = \int d^4x' \; \Delta_r(x - x')J^\mu_t \qquad (21c)$$

The effect of the tenuous plasma that fills the universe is to replace the Wheeler-Feynmann Green's function with the retarded Green's function. Note that this came about because the damping term in Eq.(16a). The effect of a positive $\nu$ was to shift poles from the real axis of the $\omega$-plane into the lower half plane. The time assymetry in electrodynamics has emerged as a consequence of the time assymetry in the equations of mechanics that we introduced when we wrote Eq.(16a) with a damping term, thus destroying its invariance under time reversal. If we had chosen $\nu$ to be negative, then poles in the Green's functions would have been shifted into the upper half $\omega$-plane and advanced instead of retarded potentials would have been obtained; in effect, past and future would be interchanged.

Let us write the Fourier transform of Eq.(21c) as

$$A^\mu = \Delta_r J^\mu{}_t = \Delta_{WF} J^\mu_t + A^\mu_i \qquad (22)$$

where $A^\mu_i$ is the potential due to currents induced in the plasma. It follows that

$$A^\mu_i = (\Delta_r - \Delta_{WF}) J^\mu{}_t$$

$$= 1/2(\Delta_r - \Delta_a) J^\mu_t \qquad (23)$$

The induced fields are the difference of one half retarded and one half advanced fields. When added to the fields that arrive directly from the test current, the advanced fields cancel and the retarded fields add, giving the full retarded fields.

The interaction of the a particle with itself has already been taken account of; it gave the electromagnetic correction to the mass of the particle of Eq.(8b). The test particle will also experience the fields produced by the induced currents that we have just found to be one half of the difference of retarded and advanced fields; namely

$$1/2(F^{\mu\nu}_{r} - F^{\mu\nu}_{a}) \qquad (24)$$

If we repeat the calculations of Chapter 8 that led to Eq.(108), we find that the force exerted by these fields is

$$\vec{F}_i = (2e^2/3c^3)\dddot{\vec{a}} \qquad (25)$$

That is, the universe responds to the test particle with a force that is just the radiation reaction that we previously found. This gives us a rather different view of the loss of energy by radiation. A particle radiates only because there are other particles in the universe to absorb the energy. If there were only one particle in the universe, that particle could not radiate. The energy lost by the radiating particle is absorbed by particles whose space-time separation from it is zero. Loosely speaking, the radiating and absorbing particles are in intimate contact, although the separation in space or in time may be arbitrarily large.

## LORENTZ INVARIANT THEORIES OF GRAVITATION

In attempting to construct a relativistic theory of gravitation, we shall be guided by the very successful theory of charged particles interacting with the electromagnetic field. Let us review this theory. We write the Lagrangian for particle a of mass $m_a$, charge $e_a$ and coordinates $a^\mu$ as

$$L = -(m_a/2)\eta_{\mu\nu}\dot{a}^\mu \dot{a}^\nu - e_a A_\mu(a^\nu)\dot{a}^\mu \tag{26}$$

The dot over a symbol denotes a derivative with respect to proper time; the differential of proper time is

$$da = (\eta_{\mu\nu} da^\mu da^\nu)^{1/2} \tag{27}$$

It is convenient to work in natural units with $c = \hbar = 1$, as this will simplify dimensional analysis. Lagrange's equations give the equations of motion

$$m_a a^\beta = e_a \eta^{\beta\mu}(A_{\alpha,\mu} - A_{\mu,\alpha})\dot{a}^\alpha \tag{28}$$

These must be supplemented by the equation

$$A_\mu(a^\nu) = \sum_{b\neq a} e_b \int db\, \dot{b}_\mu \delta(ab \cdot ab) \tag{29}$$

that tells us how the electromagnetic potential is related to the charged particles that produce it. The integral is along the world line of particle b. Performing the integration gives the Lienard-Weichert potentials

$$A_\mu(a^\nu) = \sum_{a\neq b} e_b \left[\frac{\dot{b}_\mu}{\dot{b}\cdot ab}\right]_{ab\cdot ab} \tag{30}$$

The subscript indicates that the quantity in brackets is to be evaluated at the points on the world line of b where it is intersected by the light cone from the point $a^\nu$. To get only the retarded interaction we omit the contribution from the future light cone. To get both retarded and advanced contributions we take contributions from both past and future light cones and divide by two.

We now ask ourselves why only the first power of the velocity should enter the interaction term of the Lagrangian as is the case for the electromagnetic interaction $e_a A_\mu \dot{a}^\mu$. Is it not possible for other powers of the four-velocity to appear. In trying to construct a theory of gravitation we are led to try the Lagrangian

$$L = m_a(1/2\ \eta_{\mu\nu}\dot{a}^\mu \dot{a}^\nu + \tilde{g} + \tilde{g}_\mu \dot{a}^\mu$$

$$+ 1/2\ \tilde{g}_{\mu\nu}\dot{a}^\mu \dot{a}^\nu + 1/3\ \tilde{g}_{\mu\nu\gamma}\dot{a}^\mu \dot{a}^\nu \dot{a}^\gamma + \text{etc.}) \qquad (31)$$

We have included a factor of $m_a$ in each term since we expect the gravitational force to be proportional to the mass. In the above $\tilde{g}$ is a scalar, $\tilde{g}_\mu$ is a vector and in general $\tilde{g}_{\mu\nu\gamma\ldots}$ is a covariant tensor of rank indicated by the number of indices. Without loss of generality we may assume that all of these tensor potentials are symmetric to interchange to any pair of indices. We would expect that only one of these interaction terms was non-zero; otherwise we would have to attribute gravitation to two or more fields. The dot over a symbol denotes differentiation with respect to proper time as before, but we no longer require that the differential of proper time be given by Eq.(27). For the moment proper time is undefined.

Lagrange's equations give the equations of motion

$$\eta_{\mu\nu}\ddot{a}^\nu + \Gamma_\mu + \Gamma_{\mu\nu}a^\nu + [\tilde{g}_{\mu\nu}\ddot{a}^\nu + \Gamma_{\mu\nu\alpha}\dot{a}^\nu \dot{a}^\alpha]$$

$$+ [\tilde{g}_{\mu\nu\gamma}(\dot{a}^\nu \ddot{a}^\gamma + \ddot{a}^\nu \dot{a}^\gamma) + \Gamma_{\mu\nu\gamma\alpha}\dot{a}^\nu \dot{a}^\gamma \dot{a}^\alpha] + \text{etc.} = 0 \qquad (32a)$$

where

$$\Gamma_\mu = -\tilde{g}_{,\mu} \qquad (32b)$$

$$\Gamma_{\mu\nu} = \tilde{g}_{\mu,\nu} - \tilde{g}_{\nu,\mu} \qquad (32c)$$

$$\Gamma_{\mu\nu\alpha} = 1/2[\tilde{g}_{\mu\nu,\alpha} + \tilde{g}_{\mu\alpha,\nu} - \tilde{g}_{\nu\alpha,\mu}] \qquad (32d)$$

$$\Gamma_{\mu\nu\gamma\alpha} = 1/3[\tilde{g}_{\mu\nu\gamma,\alpha} + \tilde{g}_{\mu\gamma\alpha,\nu} + \tilde{g}_{\mu\nu\alpha,\gamma} - \tilde{g}_{\nu\gamma\alpha,\mu}] \qquad (32e)$$

The Hamiltonian

$$H = \dot{a}^\mu(\partial L/\partial \dot{a}^\mu) - L$$

$$= m_a(1/2\ \eta_{\mu\nu}\dot{a}^\mu\dot{a}^\nu - \tilde{g} + 1/2\ \tilde{g}_{\mu\nu}\dot{a}^\mu\dot{a}^\nu \quad + \text{etc.}) \qquad (33)$$

is easily shown to be a constant of the motion. This may be used to define the differential of proper time da. Let us suppose that there are no gravitational potentials of higher than second rank; then we drop the terms + etc. in Eq.(33) and solve for da. If we choose $2H/m_a = 1$, we find

$$da = \left[\frac{(\eta_{\mu\nu} + \tilde{g}_{\mu\nu})da^\mu da^\nu}{1 + 2\tilde{g}}\right]^{1/2} \qquad (34)$$

We see that the proper time depends on the gravitational potentials. Our choice of $2H/m_a$ has made da reduce to Eq.(27) in the absence of gravitational potentials.

Next, we turn to the relation between the potential $\tilde{g}_{\mu\nu\gamma\ldots}$ and the massive particles that produce it. Analogy with Eq.(29) suggest that we postulate

$$\tilde{g}_{\mu\nu\gamma\ldots}(a^{\alpha}) = \lambda \sum_{b\neq a} G\, m_b \int db\, \delta(ab\cdot ab)\, K_{\mu\nu\gamma\ldots} \qquad (35)$$

where $\lambda$ is a suitably chosen numerical constant and $K_{\mu\nu\gamma\ldots}$ is some appropriately constructed completely symmetric tensor. The integration over db may be performed to obtain the analog of the Lienard-Weichert potentials with the result

$$\tilde{g}_{\mu\nu\gamma\ldots} = \lambda \sum_{b\neq a} G\, m_b \left[\frac{K_{\mu\nu\gamma\ldots}}{|\dot{b}\cdot ab|}\right]_{ab\cdot ab} \qquad (36)$$

The factor $\delta(ab\cdot ab)$ in Eq.(35) assures us that particles interact gravitationally only when their space-time separation is zero. Lorentz invariance is satisfied by requiring $K_{\mu\nu\gamma\ldots}$ and hence $\tilde{g}_{\mu\nu\gamma}$ to be tensors. The only quantities available for the construction of $K_{\mu\nu\gamma\ldots}$ are $\dot{b}_{\mu}$, $ab_{\mu} = a_{\mu} - b_{\mu}$ and $\eta_{\mu\nu}$. Inspection of Eqs.(31) and (36) shows that $g_{\mu\nu\gamma\ldots}$ and $K_{\mu\nu\gamma\ldots}$ are dimensionless. If $ab_{\mu}$ is used in the construction of $K_{\mu\nu\gamma\ldots}$, it can be made dimensionless by forming the combination $(ab_{\mu}/\dot{b}\cdot ab)$. The construction of $K_{\mu\nu\gamma\ldots}$ is further restricted by the requirement that it be symmetric to interchange of any pair of indices. Let us consider some examples:

There is only one scalar field

$$\tilde{g}(a^{\nu}) = \lambda \sum_{b\neq a} Gm_b \int db\, \Delta(ab\cdot ab). \qquad (37)$$

There are two vector fields

$$\begin{Bmatrix} \tilde{g}_{\mu}^{(1)}(a_{\nu}) \\ \tilde{g}_{\mu}^{(2)}(a_{\nu}) \end{Bmatrix} = \lambda \sum_{b\neq a} Gm_b \int db\, \Delta(ab\cdot ab) \begin{Bmatrix} \dot{b}_{\mu} \\ \dfrac{ab_{\mu}}{\dot{b}\cdot ab} \end{Bmatrix} \qquad (38)$$

The first of these choices gives a theory closely analogous to electrodynamics. There are four choices for a second-rank tensor:

$$\left\{\begin{array}{l} \tilde{g}^{(1)}_{\mu\nu}(a_\nu) \\ \tilde{g}^{(2)}_{\mu\nu}(a_\nu) \\ \\ \tilde{g}^{(3)}_{\mu\nu}(a_\nu) \\ \tilde{g}^{(4)}_{\mu\nu}(a_\nu) \end{array}\right\} = \lambda \sum_{b\neq a} Gm_b \int db\, \Delta(ab\cdot ab) \left\{\begin{array}{l} \dot{b}_\mu \dot{b}_\nu \\ \dfrac{ab_\mu ab_\nu}{(\dot{b}\cdot ab)^2} \\ \eta_{\mu\nu} \\ \dfrac{\dot{b}_\mu \cdot ab_\mu + ab_\mu \cdot \dot{b}_\nu}{\dot{b}\cdot ab} \end{array}\right\} \qquad (39)$$

There are six choices for a third-rank tensor. We shall not give any other examples, since the pattern should now be clear.

We shall require that each theory reduce to Newton's gravitational theory in the limit of small velocities and weak gravitationl fields. By small velocities we mean

$$|\dot{a}^i| \ll 1 \qquad i = 1,2,3$$
$$\dot{a}^0 \simeq 1 \qquad (40)$$

for all a. By weak gravitational fields we mean

$$|\tilde{g}_{\mu\nu\gamma..}| \ll 1 \qquad (41)$$

In this limit Eq.(31) becomes

$$L = -m_a/2 + m_a/2|d\vec{a}/dt|^2 + (m_a/n)\tilde{g}_{000..}(\vec{a},t) \qquad (42)$$

(Only one interaction term has been retained; namely, that for a tensor of rank n). This is the Lagrangian for a nonrelativistic particle moving in the potential $(m_a/n)g_{000..}(\vec{a},t)$. In this limit

$$\dot{b}\cdot ab \simeq \dot{b}^0 ab_0 \simeq ab_0 = -t_{ab} = \pm r_{ab} \tag{43a}$$

and

$$ab_0/(\dot{b}\cdot ab) \simeq 1 \tag{43b}$$

It follows that $K_{000...} = \pm 1$. Then the factor $\lambda$ may be chosen as $\pm n$ so as to give

$$(m_a/n)g_{000..} = + \sum_{b\neq a} Gm_a m_b (1/r_{ab}) \tag{44}$$

as the limit of Eq.(36). We see that each of our Lorentz invariant gravitational theories gives

$$L_a = - m_a/2 + m_a/2\ |d\vec{a}/dt|^2 + \sum_{b\neq a} Gm_a m_b (1/r_{ab}) \tag{45}$$

for the Lagrangian of particle a in the small-velocity-weak-field limit. This agrees with Newton's theory.

We are faced with an embarassment of riches. How shall we choose from among this infinite number of Lorentz-invariant theories, all of which agree with Newton's theory in the lowest approximation? Clearly, we must appeal to observations to cast the deciding vote. The predictions of small deviations from Newton's theory must be looked for and compared with observations. Planetary motion seems to be a good place to look.

We shall calculate the orbit of a planet about the sun, which is assumed to be fixed at the origin of the coordinate system. This will be done for only one of the theories as an example, and then we shall quote the results for the other cases we have considered. For our example we shall choose $\tilde{g}^{(2)}_{\mu\nu}$. This is of historical interest since it is a theory of gravitation proposed by the mathematician and philosopher Alfred North Whitehead in 1922. Since the sun is fixed, $\dot{b}_0 = 1$ and $\dot{b}_1 = \dot{b}_2 = \dot{b}_3 = 0$. Eq.(39) gives

$$\tilde{g}^{(2)}_{\mu\nu} = (2GM/r) \begin{cases} 1 & \mu = \nu = 0 \\ (x^i x^j/r^2) & \mu = i = 1,2,3 \\ & \nu = j = 1,2,3 \\ -(x^i/r) & \mu = 0 \\ & \nu = i = 1,2,3 \\ -(x^j/r) & \mu = j = 1,2,3 \\ & \nu = 0 \end{cases} \quad (46)$$

The orbit may be shown to lie in a plane as in the nonrelativistic theory. In polar coordinates the Lagrangian is found to be

$$L = -(\dot{t}^2/2)(1 - 2GM/r) + (\dot{r}^2/2)(1 + 2GM/r)$$

$$+ \dot{r}^2\dot{\phi}^2/2 - (2GM/r)\dot{r}\dot{t} \quad (47)$$

(Since the orbit is independent of the mass of the planet, we have chosen the mass to be unity.) Since L is independent of t, $\phi$ and the proper time, three constants of the motion are immediately found; these are.

$$p_t = \partial L/\partial\dot{t} = -\dot{t}(1 - 2GM/r) - (2GM/r)\dot{r} \quad (48a)$$

$$p_\phi = \partial L/\partial\phi^{\cdot} = r^2\dot{\phi} \quad (48b)$$

$$E = p_t\dot{t} + p_r\dot{r} + p_\phi\dot{\phi} - L = L \quad (48c)$$

Eq.(48a) and (48b) may be used to eliminate $\dot{t}$ and $\dot{\phi}$ from E. Then the independent variable is changed from proper time to $\phi$ by using

$$\dot{r} = \dot{\phi}(dr/d\phi) = (p_\phi/r^2)dr/d\phi = -p_\phi(du/df) \quad (49a)$$

where

$$u = 1/r \quad (49b)$$

We find

$$(du/d\phi)^2 + u^2(1 - 2GMu) + (4GME/p_\phi{}^2)u = (2E + p_t{}^2) \quad (50a)$$

Differentiating this equation gives

$$d^2u/d\phi^2 + u + 2GME/p_\phi{}^2 = 3GMu^3 \quad (50b)$$

We shall solve this equation approximately by assuming that the orbit is nearly circular. We write

$$u = u_0 + u_1(\phi) \qquad |u_1| << u_0 \quad (51)$$

where $u_0$ is constant and satisfies

$$u_0 + 4GME/p_\phi{}^2 = 3GMu_0{}^3 \quad (52a)$$

$$u_0 \simeq - 4GME/p_\phi{}^2 \equiv 1/a(1 - e^2) \quad (52b)$$

and $u_1$ satisfies

$$d^2u_1/d\phi^2 + u_1 = 3GM(2u_0u_1 + u_1{}^2) \simeq 6GMu_0u_1 \quad (53)$$

Thus we find approximate solutions of Eq.(50) to be

$$u = 1/r = \frac{1 + e\cos\omega\phi}{a(1 - e^2)} \quad (54a)$$

where

$$\omega = (1 - \frac{6GM}{a(1 - e^2)})^{1/2} \simeq 1 - \frac{3GM}{a(1 - e^2)} \quad (54b)$$

Eq.(44) has the form of a precessing ellipse of major radius a and eccentricity e. The minimum value of r occurs at $\phi = 0$ and again at

$\phi = 2\pi/\omega$. Therefore the precession per orbit is

$$\delta\phi = 2\pi(1/\omega - 1) \simeq 6GM/a(1 - e^2) \tag{55}$$

This is the same precession as that predicted by the general theory of relativity, which, as is well known, is in good agreement with observation.

The calculation of planetary precession for the other theories proceeds as in the example. In each case the three constants of the motion of Eq.(38) are found. The calculations become difficult for third- or higher- rank theories, so in the absence of strong motivations we have limited our calculations to scalar, vector and second-rank tensor theories. The scalar theory of Eq.(37) predicts no precession. The two vector theories of Eq.(38) give the same result; namely, 1/6 of the value predicted by general relativity. The four second-rank tensor theories of Eq.(39) predict precessions that differ from that of general relativity by factors of 2/3, 1, -8/3 and -2/3 respectively. A particular linear combination of $g_{\mu\nu}^{(1)}$ and $g_{\mu\nu}^{(3)}$ is also of interest. This is

$$g_{\mu\nu}(lgr) = 4\sum_{b\neq a} Gm_b \int db\, \Delta(ab\cdot ab)(\dot{b}_\mu \dot{b}_\nu - \eta_{\mu\nu}\dot{b}^\sigma \dot{b}_\sigma/2) \tag{56}$$

This is the linearized theory of gravitation obtained from the general theory of relativity by writing

$$g_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu}(lgr), \tag{57a}$$

and supposing

$$|g_{\mu\nu}(lgr)| << 1 \tag{57b}$$

This theory predicts a precession greater than that predicted by general relativity by a factor of 4/3.

According to Eq.(34) the scalar and second-rank tensor theories (but not the vector theories) predict a gravitational time dilation. For a body at rest in a gravitational field $da^0 = 1$ and $da^i = 0$

for i = 1,2,3. For Whitehead's theory $g_{00}^{(2)} = 2GM/r$, and

$$da = dt(1 + 2GM/r)^{1/2} \tag{58}$$

This also agrees with the predictions of general relativity and with observations. It has also been shown that Whitehead's theory predicts a bending of a light ray in a gravitational field that agrees with general relativity and with observations. The three classical test of general relativity are the precession of planetary orbits, gravitational time dilation and the bending of light rays in a gravitational field. Whitehead's theory and general relativity agree with one another and with observations. It was not until 1971 that it was shown that Whitehead's theory predicted tides on the earth with a 12-hour sidereal period due to the mass of our galaxy that would have an amplitude that was 200 times greater than the upper limit of sensitive gravimeter measurements. By this time the philosophical and aesthetic appeal of general relativity together with its essentially perfect agreement with observations had won for it almost universal acceptance as the correct theory of gravitation.

# CHAPTER 10

# THE GENERAL THEORY OF RELATIVITY

## "DERIVATION" OF EINSTEIN'S EQUATIONS

The general theory of relativity formulated by Albert Einstein in 1915 is based on two general principles, The Principle of General Covariance and the Principle of Equivalence. The first of these is the requirement that the laws of physics should be independent of the reference system. It should be possible to write the laws of physics in a form that is valid in all spacetime coordinate systems including accelerated ones. This requirement is met by writing the laws of physics as tensor equations. Under a transformation of coordinates all terms in an equation transform in the same way, so if the equation is satisfied in one coordinate system it is satisfied in all. We accomplished this in Chapter 8 by writing the equations of motion for a particle in an electromagnetic field as

$$m[d^2x^\mu/d\tau^2 + \Gamma^\mu_{\alpha\beta}(dx^\alpha/d\tau)(dx^\beta/d\tau)] = e/c\ F^\mu{}_\nu(dx^\nu/dt) \qquad (1)$$

and Maxwell's equations as

$$F^{\mu\nu}{}_{;\nu} = -\ 4\pi/c\ J^\mu \qquad (2a)$$

$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0 \qquad (2b)$$

The Principle of Equivalence is suggested by the experimental determination of the equivalence of inertial and gravitational mass. The gravitational force on a body is proportional to a property of the body that we call the gravitational mass. In Newton's second law of motion the force is equated to the acceleration of the body multiplied by a property of the body that we call the inertial mass.

Experiments have shown that these two masses are proportional to a very high degree of accuracy regardless of the size or composition of the body. With a proper choice of units the coefficient of proportionality can be set equal to unity and then the two masses are equal. All bodies fall in precisely the same way in a gravitational field since the mass cancels from the two sides of the equations of motion. Inertial forces such as centrifugal, Coriolis and other "fictitious" forces that appear in the equations of motion due to a choice of an accelerated coordinate system share with gravitational force the property of being proportional to mass. The inertial forces appear in Eq.(1) in the term containing $\Gamma$. They may be "transformed away" by making a transformation of coordinates to an inertial system with a constant metric tensor $g_{\mu\nu}$. This suggest that gravitational forces may also be included in the $\Gamma$-term, and then we may regard inertial and gravitational forces as equivalent, at least locally. We can always choose a coordinate system in which the $\Gamma$-term vanishes in a sufficiently small region about a spacetime point p. To show this let us recall the transformation formula for $\Gamma$

$$\Gamma^{\iota}_{\kappa\lambda} = \Gamma^{\mu'}{}_{\nu'\rho'} x^{\iota}{}_{,\mu'} x^{\nu'}{}_{,\kappa} x^{\rho'}{}_{,\lambda} + x^{\mu'}{}_{,\kappa,\lambda} x^{\iota}{}_{,\mu'} \qquad (3)$$

We shall choose a new set of coordinates given in terms of the old by

$$x^{\iota'} = x^{\iota} + 1/2\, (\Gamma^{\iota}{}_{\kappa\lambda})_p (x^{\kappa} - x^{\kappa}_p)(x^{\lambda} - x^{\kappa}_p) \qquad (4)$$

where the subscript p denotes evaluation at the spacetime point p. Calculating the derivatives and substituting into Eq.(3) gives $(\Gamma^{\mu'}{}_{\nu'\rho'})_p = 0$. A coordinate system in which $\Gamma$ vanishes at a point is said to be locally-inertial.

We may formulate the Weak Principle of Equivalence as the statement that "at every spacetime point p in an arbitrary gravitational field it is possible to choose a locally-inertial coordinate system such that, within a sufficiently small region about p, the laws of motion of freely falling particles take the same form as in unaccelerated Cartesian coordinate systems". The Strong Principle of Equivalence goes further and replaces the words "laws of motion of freely falling particles" with "all of the laws of physics". This implies that the special theory of relativity is applicable in any sufficiently small space time region. In order to write the laws of physics in a form that is valid in the presence of a gravitational field, one writes them in the form that is valid in the absence ~~in the absence~~ of a gravitational field but in generally covariant form. Thus, Eqs.(1) and (2) express the laws of particle mechanics and electrodynamics in the absence or in the presence of a gravitational field.

Let us see qualitatively what some of the implications of the Principle of Equivalence are by comparing the observations of physical phenomena in two laboratories, one of which is at rest in a uniform gravitational field with gravitational acceleration g, and the other in an elevator that is accelerated upward with acceleration g. This is depicted in Fig.1.

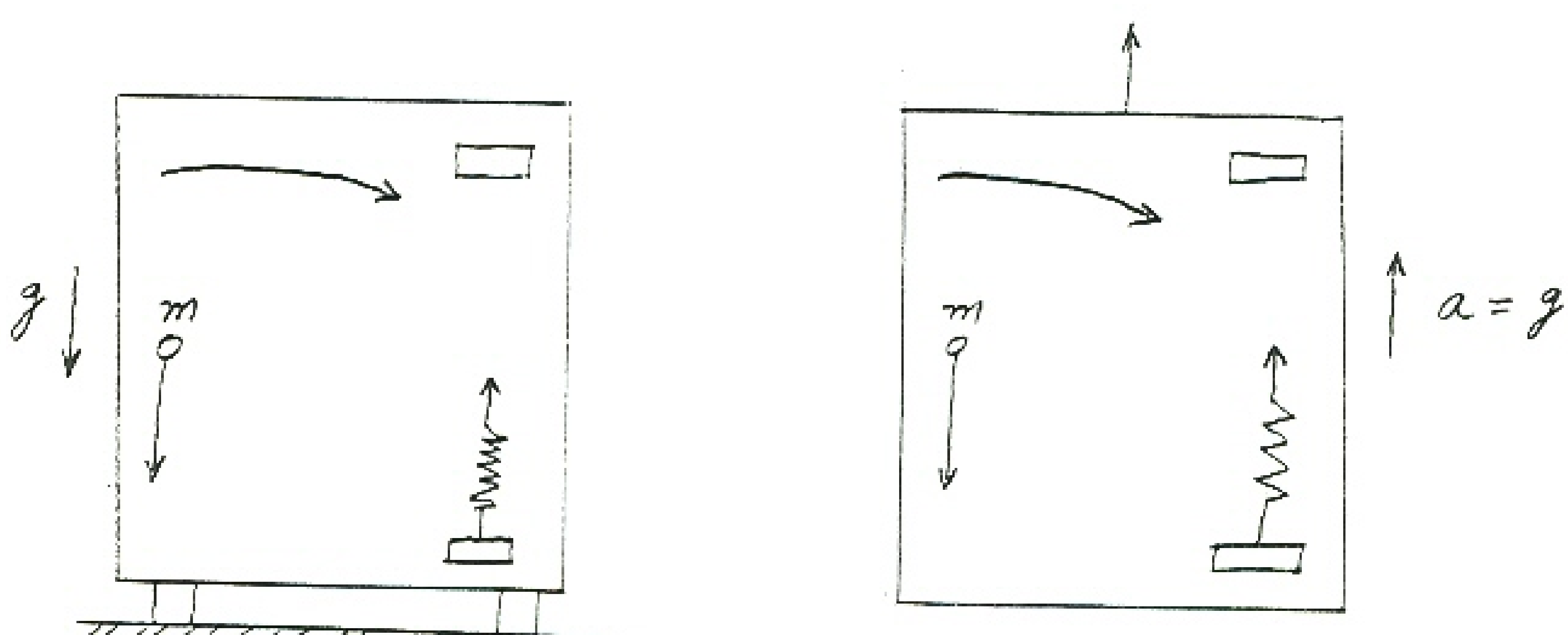


Fig.1

If a ball of mass m was dropped in each laboratory, an observer in each would observe its apparent accelleration toward the floor with acceleration g. The first observer would interpret this as being due to a force mg exerted on the particle by the gravitational field, while the second observer would say that his ball remained stationary while the floor accelerated toward it. Both observers see the same acceleration g according to the Weak Principle of Equivalence. If gravitational and inertial masses were not equal, the observer in the elevator would see the same acceleration for all bodies dropped in his laboratory, but the observer in the gravitational field would see different accelerations depending on the size or composition of the bodies. A ray of light moving in a straight line would appear to move in a curved path to the observer in the elevator because of the acceleration of the laboratory floor toward the path. According to the Strong Principle of Equivalence, the observer in the gravitational field must also observe a curved path for a light ray in his laboratory. Thus, the Strong Principle of Equivalence allows us to infer the bending of the path of a light ray in a gravitational field. Next, consider an experiment in each laboratory in which there is a source of light near the floor that is observed with a spectrometer near the ceiling. The observer in the elevator would find the wavelength of the light to be shifted toward the red, because, during the time $t = h/c$ it took for the light to traverse the distance h between floor and ceiling, the spectrometer had increased its velocity by $v = gt = gh/c$. As a result there is a Doppler shift of $\Delta\nu/\nu = v/c = gh/c^2$. According to the Strong Principle of Equivalence the observer in the gravitational field should see the same red shift which can be written as $\Delta\nu/\nu = \Delta\Phi/c^2$ where $\Delta\Phi = gh$ is the difference in gravitational potential between floor and ceiling. Thus we infer the gravitational red shift.

Neither observer would be able to infer from measurements made in his laboratory whether he was in a gravitational field or in an accelerated elevator. However, a nonuniform gravitational field could be distinguished from an accelerated system. For instance, in a laboratory on the surface of the earth, if two balls were dropped simultaneously, they would approach each other as they fell to the floor because they would fall toward the center of the earth. True gravitational fields are distinguished from fictitious gravitational fields due to accelerated reference systems by our inability to abolish them everywhere by a coordinate transformation; they can only be abolished over a sufficiently small spacetime region. Mathematically, this means that we cannot find a transformation to a coordinate system with a constant metric tensor, and this in turn implies that the curvature tensor $R_{\mu\nu\alpha\beta}$ is nonvanishing. Gravitation is to be explained by the curvature of spacetime

There remains the problem of discovering the equations that relate the spacetime curvature to the distribution of mass, momentum and energy. We shall be guided by Newton's theory of gravitation. We write the nonrelativistic equations of motion for a particle in a gravitational field as

$$m d^2x^i/dt^2 = - m \phi_{,i} \tag{5}$$

where $\phi$ is the gravitational potential which is related to the mass density $\rho$ by

$$\nabla^2 \phi = 4\pi G\rho \tag{6}$$

We know that these equations give an accurate discription of gravitation for sufficiently small velocities and weak gravitational fields. For small velocities we may neglect the space-like components of the particles 4-velocity and set the time-like component equal to c in the $\Gamma$-term of Eq.(1) which becomes

$$d^2x^i/dt^2 + \Gamma^i_{00}\, c^2 = 0 \tag{7}$$

Comparing this with Eq.(5) we identify

$$\phi_{,i} = c^2\, \Gamma^i_{00} \tag{8}$$

For almost flat spacetime and time-independent gravitational field

$$\Gamma^i_{00} = 1/2\ \eta^{i\beta}(g_{\beta 0,0} + g_{\beta 0,0} - g_{00,\beta})$$

$$= 1/2\ g_{00,i} \tag{9}$$

Using this in Eq.(8) and integrating we obtain

$$g_{00} = 1 + 2\Phi/c^2 \qquad (10)$$

When this is used in Eq.(6) we obtain

$$\nabla^2 g_{00} = (8\pi G/c^2)\rho \qquad (11)$$

We wish to generalize this to a tensor equation. The mass density $\rho$ is not a scalar but is contained in the energy momentum tensor

$$T_{\mu\nu} = (\rho + \rho U/c^2 + P/c^2)u_\mu u_\nu - g_{\mu\nu}P \qquad (12a)$$

and so

$$T_{00} \simeq (\rho + \rho U/c^2)c^2 \simeq \rho c^2 \qquad (12b)$$

We use this to rewrite Eq.(11) as

$$\nabla^2 g_{00} = (8\pi G/c^4)T_{00} \qquad (13)$$

Eq.(13) suggest that the field equations of general relativity that we are searching for have the form

$$G_{\mu\nu} = -(8\pi G/c^4)T_{\mu\nu} \qquad (14)$$

where $G_{\mu\nu}$ is a second rank tensor with the properties:

(A) $G_{\mu\nu} = G_{\nu\mu}$ since $T_{\mu\nu} = T_{\nu\mu}$.

(B) Linear in th second derivatives of $g_{\mu\nu}$.

(C) $G^{\mu\nu}{}_{;\nu} = 0$ since $T^{\mu\nu}{}_{;\nu} = 0$

(D) $G_{00} \rightarrow + \Box^2 g_{00}$ for weak fields.

$- \rightarrow +$

Our first guess might be that $G_{\mu\nu}$ is the generalization of $\frac{1}{2}\Box^2 g_{\mu\nu}$ obtained by replacing the derivatives in the D'Alembertian by covariant derivatives. However, this will not work since the covariant derivative of the metric tensor vanishes; $G_{\mu\nu}$ would be identically zero.

The curvature tensor $R^{\alpha}{}_{\beta\mu\nu}$ is linear in the second derivatives of $g_{\mu\nu}$, but it cannot be a candidate since it is of fourth rank. Contracting it gives the Ricci tensor $R_{\beta\nu} = R^{\alpha}{}_{\beta\alpha\nu}$. This is a promising candidate since it is second rank, symmetric and linear in the second derivatives of $g_{\mu\nu}$. To investigate whether property (C) is satisfied we shall make use of the Bianchi identities

$$R^{\lambda}{}_{\mu\nu\kappa;\eta} + R^{\lambda}{}_{\mu\eta\nu;\kappa} + R^{\lambda}{}_{\mu\kappa\eta;\nu} = 0 \qquad (15)$$

which we shall now prove. It will be recalled that the curvature tensor has the form

$$R = \partial\Gamma/\partial x - \partial\Gamma/\partial x + \Gamma\Gamma - \Gamma\Gamma \qquad (16)$$

In a locally-inertial coordinate system the $\Gamma$'s vanish, so

$$R^{\lambda}{}_{\mu\nu\kappa;\eta} = (\partial/\partial x^{\eta})[\partial\Gamma^{\lambda}{}_{\mu\nu}/\partial x^{\kappa} - \partial\Gamma^{\lambda}{}_{\mu\kappa}/\partial x^{\nu}] \qquad (17)$$

Calculating the other two terms in Eq.(15) we find that all terms cancel, so the Bianchi identities are satisfied in a locally-inertial system. But Eq.(15) is a tensor equation, so if it is satisfied in one coordinate system, it is satisfied in all. This completes the proof.

Now, we contract Eq.(15) by setting $\kappa = \lambda$ and obtain

$$-R_{\mu\nu;\eta} + R^{\lambda}{}_{\mu\eta\nu;\lambda} + R_{\mu\eta;\nu} = 0 \qquad (18)$$

We have used the antisymmetry of R in the last pair of indices. Next, we multiply by $g^{\mu\nu}$ and contract to obtain

$$- R_{;\eta} + 2R^{\lambda}{}_{\eta;\lambda} = 0 \tag{19}$$

Finally, we multiply by $g^{\sigma\eta}$ and contract to obtain

$$(R^{\sigma\eta} - 1/2\, g^{\sigma\eta}R)_{;\eta} = 0 \tag{20}$$

We see that $R^{\sigma\eta}$ is not a satisfactory choice, since $R^{\sigma\eta}{}_{;\eta} \neq 0$ but that

$$G^{\sigma\eta} = R^{\sigma\eta} - 1/2 g^{\sigma\eta} R \tag{21}$$

does satisfy requirements (A), (B) and (C),

It remains to be shown that requirement (D) is also satisfied. This will be shown in the next section, but in anticipation of this demonstration we write the Einstein field equations as

$$G_{\mu\nu} = R_{\mu\nu} - 1/2\, g_{\mu\nu}\, R = -(8\pi G/c^4)T_{\mu\nu} \tag{22}$$

Note that contraction gives

$$R = R^{\nu}{}_{\nu} = (8\pi G/c^4)T^{\nu}{}_{\nu} = (8\pi G/c^4)T \tag{23}$$

so the field equations can also be written as

$$R_{\mu\nu} = -(8\pi G/c^4)(T_{\mu\nu} - 1/2\, g_{\mu\nu}T) \tag{24}$$

## THE LINEARIZED FIELD EQUATIONS

We shall assume that the metric tensor $g_{\mu\nu}$ differs very little from the Lorentz metric tensor $\eta_{\mu\nu}$ and write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \qquad |h_{\mu\nu}| << 1 \tag{25}$$

In calculating the connection coefficients $\Gamma^{\lambda}_{\mu\nu}$ and Ricci tensor $R_{\mu\nu}$ we shall retain only terms that are linear in $h_{\mu\nu}$. We find

$$\Gamma^{\lambda}_{\mu\nu} = 1/2 \; \eta^{\lambda\rho}[\partial h_{\mu\rho}/\partial x^{\nu} + \partial h_{\nu\rho}/\partial x^{\mu} - \partial h_{\mu\nu}/\partial x^{\rho}] \tag{26a}$$

$$R_{\mu\nu} = \partial \Gamma^{\lambda}_{\lambda\mu}/\partial x^{\nu} - \partial \Gamma^{\lambda}_{\mu\nu}/\partial x^{\lambda}$$

$$= 1/2 \; (\Box^2 h_{\mu\nu} + \partial_{\nu}\partial_{\mu} h^{\lambda}_{\lambda} - \partial_{\nu}\partial_{\lambda} h^{\lambda}_{\ \mu} - \partial_{\lambda}\partial_{\mu} h^{\lambda}_{\nu}) \tag{26b}$$

We can use our freedom to make coordinate transformations to eliminate unwanted terms in Eq.(26b). Let us make a coordinate transformation

$$x^{\mu} \rightarrow x^{\mu\prime} = x^{\mu} + \xi^{\mu}(x) \tag{27}$$

where $\xi^{\mu}(x)$ is of order $h_{\mu\nu}$. A brief calculation gives

$$h_{\mu\prime\nu\prime} = h_{\mu\nu} - \partial\xi_{\mu}/\partial x^{\nu} - \partial \xi_{\nu}/\partial x^{\mu} \tag{28}$$

Let us choose $\xi^{\mu}(x)$ so that $h_{\mu\nu}$ obeys

$$\partial_{\nu}\psi^{\nu}_{\lambda} = 0 \tag{29a}$$

where

$$\psi^{\nu}_{\ \lambda} = h^{\nu}_{\ \lambda} - 1/2 \; \delta^{\nu}_{\ \lambda} h^{\mu}_{\ \mu} \tag{29b}$$

To prove that this is possible, let us suppose that in some coordinate system $\partial_\nu \psi^\nu{}_\lambda \neq 0$. Then we make the coordinate transformation of Eq.(27) and use Eq.(28) to calculate

$$\psi^{\nu'}{}_{\lambda'} = \psi^\nu_\lambda - \partial_\lambda \xi^\nu - \partial^\nu \xi_\lambda + \delta^\nu{}_\lambda \partial_\alpha \xi^\alpha \qquad (30)$$

and so

$$\partial_{\nu'} \psi^{\nu'}{}_{\lambda'} = \partial_\nu \psi^\nu{}_\lambda - \Box^2 \xi_\lambda \qquad (31)$$

We need only choose $\xi_\lambda$ to satisfy the inhomogenous D'Alembert equation

$$\Box^2 \xi_\lambda = \partial_\nu \psi^\nu{}_\lambda \qquad (32)$$

and we know how to solve this equation.

Now, we use Eq.(29) in Eq.(26b) and find that the second term is cancelled by the third and fourth terms and we are left with

$$R_{\mu\nu} = 1/2\, \Box^2 h_{\mu\nu} \qquad (33)$$

When Eq.(33) is used in Eq.(24) we obtain the linearized gravitational field equations

$$\Box^2 h_{\mu\nu} = -(16\pi G/c^4)\,(T_{\mu\nu} - 1/2\, \eta_{\mu\nu} T) \qquad (34)$$

Now,

$$T = \rho c^2 + \rho U - 3P \simeq \rho c^2 \simeq T_{00} \qquad (35)$$

When this is used in Eq.(34) we obtain

$$\Box^2 h_{00} = -(8\pi G/c^2)\rho \qquad (36)$$

which agrees with Eq.(11) for static fields. Thus the linearized field equations agree with Newton's theory in this approximation.

Eq.(34) is the inhomogenous D'Alembert equation. Its solution is

$$h_{\mu\nu}(\vec{x},t) = -(4G/c^4)\int \frac{d^3x' \; S_{\mu\nu}(\vec{x}',t_r)}{|\vec{x} - \vec{x}'|} \qquad (37a)$$

where

$$S_{\mu\nu} = T_{\mu\nu} - 1/2\ \eta_{\mu\nu}T \qquad (37b)$$

and

$$t_r = t - |\vec{x} - \vec{x}'|/c \qquad (37c)$$

is the retarded time.

## GRAVITATIONAL WAVES

In the absence of sources (that is, the energy-momentum tensor vanishes) the departure of the metric tensor from the Lorentz metric tensor obeys the equations

$$R_{\mu\nu} = 1/2\, \Box^2 h_{\mu\nu} = 0 \qquad (38a)$$

$$\partial_\nu \psi^\nu{}_\lambda = \partial_\nu [h^\nu{}_\lambda - 1/2\, \delta^\nu{}_\lambda h^\nu{}_\lambda] = 0 \qquad (38b)$$

These equations have solutions that represent plane waves propogating with the speed of light; namely

$$h_{\mu\nu} = \epsilon_{\mu\nu}\, e^{ikx} + \epsilon_{\mu\nu}{}^*\, e^{-ikx} \qquad (39)$$

where $k^\mu = (\omega/c, \vec{k})$ and $kx = k_\mu x^\mu = \omega t/c - \vec{k}\cdot\vec{x}$. The symmetric tensor $\epsilon_{\mu\nu}$ is called the polarization tensor. When Eq.(39) is substituted into Eqs.(38), we find

$$k_\mu k^\mu = \omega^2/c^2 - k^2 = 0 \qquad (40a)$$

$$k_\nu \epsilon^\nu{}_\lambda - 1/2\, k_\lambda \epsilon^\mu{}_\mu = 0 \qquad (40b)$$

The relation $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}$ reduces the number of independent components to 10. For a given propagation vector $\vec{k}$, the four equations of Eq.(40b) reduce the number of independent components to 10 - 4 = 6. We are still free to make coordinate transformations of the form of Eq.(27). According to Eq.(31), if $\xi_\lambda$ satisfies $\Box^2 \xi_\lambda = 0$, then Eq.(38b) will continue to be satisfied. Such a solution is

$$\xi_\mu(x) = i\xi_\mu\, e^{ikx} - i\xi_\mu{}^*\, e^{-ikx} \qquad (41)$$

Substituting this into Eq.(28) we find that the new polarization tensors are

$$\epsilon_{\mu'\nu'} = \epsilon_{\mu\nu} + \xi_\mu k_\nu + \xi_\nu k_\mu \qquad (42)$$

Since there are four choices for $\xi_\mu$, there must be only 10 - 4 - 4 = 2 physically significant components of the polarization tensor.

Problem 1.

(a) Consider a gravitational wave propagating in the z-direction so that $k_\mu = (k,0,0,-k)$. Use Eq.(40b) to express $\epsilon_{0i}$ and $\epsilon_{22}$ in terms of the remaining 6 components. Then choose $\xi_\mu$ so that the polarization tensor takes the form

$$\epsilon_{\mu\nu} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_{11} & \epsilon_{12} & 0 \\ 0 & \epsilon_{12} & -\epsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \qquad (43)$$

(b) Consider a rotation about the z-axis through an angle $\theta$. The polarization tensor transforms as

$$\epsilon_{\mu'\nu'} = R_{\mu'}{}^{\alpha} R_{\nu'}{}^{\beta}\, \epsilon_{\alpha\beta} \qquad (44)$$

where the components of the Lorentz transformation are

$$R_{1'}{}^{1} = R_{2'}{}^{2} = \cos\theta$$

$$R_{1'}{}^{2} = -R_{2'}{}^{1} = \sin\theta$$

$$R_{0'}{}^{0} = R_{3'}{}^{3} = 1$$

and all others vanish. Show that

$$\epsilon'_\pm = e^{\pm 2i\theta}\epsilon_\pm \qquad (45a)$$

where

$$\epsilon_\pm = \epsilon_{11} + i\epsilon_{12} \qquad (45b)$$

In general, when the polarization P of a plane wave is transformed into

$$P' = e^{ih\theta} P \tag{46}$$

by a rotation through an angle $\theta$ about the direction of propagation, it is said to have <u>helicity h</u>. We see that gravitational waves have only two physically significant polarizations, and that these have helicity $\pm$ 2.

ACTION PRINCIPLES

One of the most beautiful results of theoretical physics is that all of the laws of classical physics may be derived from an action principle. That is, the fields and particle trajectories must evolve in such a way that a certain function, the action, is an extremum. Not only does the action lead to the equations of motion of particles and fields, but an analysis of its invariances leads to conservation laws. In addition the action provides the most elegant transition between classical and quantum physics through the use of the Feynman path integral.

Let us review the procedure. Suppose we are dealing with a set of fields $\phi^a(x)$ where the index $a = 1,2,3,$----N labels the field, and we are given a function L, called the Lagrangian density, of the $\phi^a$'s and their spacetime derivatives. We define the action S to be

$$S = \int d^4x \; L(\phi^a, \partial_\mu \phi^a) \qquad (47)$$

The integration is over a certain volume of spacetime with fixed boundaries. When the set of fields $\phi^a(x)$ is specified, then S is a number. We say that S is a functional of $\phi^a$. Now, suppose we change $\phi^a(x)$ by an infinitesimal amount at each spacetime point; that is $\phi^a(x) \rightarrow \phi^a(x) + \delta\phi^a(x)$. Then $S \rightarrow S + \delta S$ where

$$\delta S = \int d^4x ((\partial L / \phi^a) \; \delta\phi^a + (\partial L/\partial \partial_\mu \phi^a) \quad \delta \partial_\mu \phi^a) \qquad (48)$$

(Summation over the repeated index a is understood). We may interchange the operations $\delta$ and $\partial_\mu$ in the last term and integrate by parts to obtain

$$\delta S = \int d^4x (\partial L/\partial \phi^a - \partial_\mu (\partial L/\partial \partial_\mu \phi^a))\delta\phi^a + \text{surface terms} \qquad (49)$$

We shall assume that $\delta\phi^a(x)$ vanishes on the boundaries of the volume of integration, so the surface terms vanish. We call

$$\delta S/\delta\phi^a(x) = \partial L/\partial\phi^a - \partial_\mu(\partial L/\partial\partial_\mu\phi^a) \qquad (50)$$

the functional derivative of S with respect to the field $\phi^a(x)$. If S is to be an extremum (either a minimum or a maximum), $\delta S$ must vanish to first order in the variations in the fields. Setting the functional derivatives equal to zero in Eq.(50) give the field equations, called the Euler-Lagrange equations, for the system.

The trick is, given the equations of motion, to find the action S whose Euler-Lagrange equations are those equations of motion. In Chap.8 we found an action, Eq.(79), that yielded Maxwell's equations for the electromagnetic field and the equations of motion for charged particles. In this section we shall add a term to this action that will yield the gravitational field equations in addition. One of the most fruitful uses of action principles is in the invention of new theories. Instead of trying to guess new equations of motion, one tries to guess a new action S.

In order for S to be an invariant, L must be a scalar density of the form $\sqrt{(-g)}\Lambda(\phi^a, \partial_\mu\phi^a)$ where $\Lambda$ is a scalar. Then the combination $\sqrt{(-g)}d^4x$ is an invariant volume element.

What shall we choose for the scalar $\Lambda$ in the action for the gravitational field? The fields of interest are the components of the metric tensor $g_{\mu\nu}$. The scalar that immediately comes to mind is the curvature scalar R. However, a difficulty is apparent. R contains not only $g_{\mu\nu}$ and its first derivatives but its second derivatives as well. Fortunately this difficulty is not fatal. After some tedious algebra using the formulas

$$\Gamma^\kappa_{\kappa\lambda} = (1/\sqrt{(-g)})\,\partial_\lambda\sqrt{(-g)} \qquad (51a)$$

$$g^{\kappa\lambda}\Gamma^\iota{}_{\kappa\lambda} = -\,(1/\sqrt{(-g)})\,\partial_\kappa(\sqrt{(-g)}\; g^{\iota\kappa}) \qquad (51b)$$

$$g_{\iota\lambda}\,\partial_\mu g^{\iota\kappa} = -\,g^{\iota\kappa}\,\partial_\mu\, g_{\iota\lambda} \qquad (51c)$$

we can show that

$$\sqrt{(-g)}\; R = \partial_\nu(\sqrt{(-g)}\; W^\nu) + \sqrt{(-g)}\; G \qquad (52a)$$

where

$$W^{\nu} = g^{\iota\nu}\Gamma^{\lambda}_{\iota\lambda} - g^{\iota\kappa}\Gamma^{\nu}_{\iota\kappa} \tag{52b}$$

$$G = g^{\iota\kappa}(\Gamma^{\lambda}_{\iota\kappa}\Gamma^{\alpha}_{\kappa\lambda} - \Gamma^{\alpha}_{\kappa\lambda}\Gamma^{\lambda}_{\iota\alpha}) \tag{52c}$$

The first term in Eq.(52a) gives surface terms when integrated. Thus

$$\int d^4x \sqrt{(-g)}\ R = \int d^4x \sqrt{(-g)}\ G + \text{surface terms} \tag{53a}$$

and so

$$\delta \int d^4x \sqrt{(-g)}\ R = \delta \int d^4x \sqrt{(-g)}\ G \tag{53b}$$

since the variation of the surface terms vanishes. R is equivalent to G in a variational principle. In fact, we can always add a divergence to a Lagrangian density without changing the Euler-Lagrange equations, since the divergence can be integrated to give surface terms that contribute nothing.

We now define the gravitational action as

$$S_g = (c^3/16\pi G) \int d^4x \sqrt{(-g)}\ R \tag{54a}$$

Its variation is

$$\begin{aligned} \delta S_g &= (c^3/16\pi G) \int d^4x\ (\delta\sqrt{(-g)}\ R + \sqrt{(-g)}\ R_{\mu\nu}\delta g^{\mu\nu} \\ &\quad + \sqrt{(-g)}\ g^{\mu\nu}\delta R_{\mu\nu}) \\ &= (c^3/16\pi G) \int d^4x \sqrt{(-g)}((R_{\mu\nu} - 1/2\ g_{\mu\nu}R)\delta g^{\mu\nu} \\ &\quad + g^{\mu\nu}\ \delta R_{\mu\nu}) \end{aligned} \tag{54b}$$

where we have used $R = g^{\mu\nu} R_{\mu\nu}$ and

$$\delta\sqrt{(-g)} = -1/2\ \sqrt{(-g)}\ g_{\mu\nu}\ \delta g^{\mu\nu} \tag{54c}$$

The term in the integrand containing $\delta R_{\mu\nu}$ can be written as a divergence and discarded, as we shall now show. We find

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\lambda} - \delta\Gamma^{\lambda}_{\mu\nu} + \delta\Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} + \Gamma^{\eta}_{\mu\lambda}\delta\Gamma^{\lambda}_{\nu\eta} - \delta\Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} - \Gamma^{\eta}_{\mu\nu}\delta\Gamma^{\lambda}_{\lambda\eta} \tag{55}$$

This may be written as

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (\delta\Gamma^{\lambda}{}_{\mu\nu})_{;\lambda} \tag{56}$$

This relation is known as the <u>Palatini identity</u>. In writing this equation we have treated $\delta\Gamma$ as if it were a tensor. Is this justified? The answer is affirmative, for consider the transformation formula for $\Gamma$, Eq.(3). In taking the difference of two $\Gamma$'s to obtain $\delta\Gamma$, the inhomogenous term cancels, so $\delta\Gamma$ transforms as a tensor. Although $\Gamma$ is not a tensor $\delta\Gamma$ is.

Now we may write

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}[(\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}] \\ &= [(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda}] \\ &= [g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\lambda}]_{;\nu} \\ &= \delta V^{\nu}{}_{;\nu} = (1/\sqrt{(-g)})\partial_{\nu}(\sqrt{(-g)}\ \delta V^{\nu}) \end{aligned} \tag{57}$$

The justification for taking $g^{\mu\nu}$ inside the parentheses in the second line is that the covariant derivatives of $g^{\mu\nu}$ vanish, so $g^{\mu\nu}$ behaves as a constant for covariant derivations. The vector $V^{\nu}$ is the quantity in square brackets in the third line. In the final step we have used the formula for the divergence of a vector.

Now, putting Eq.(57) into (54b) we integrate to obtain surface terms that vanish, and we find for the functional derivative

$$\delta S_g/\delta g_{\mu\nu} = (c^3/16\pi G)\sqrt{(-g)}\ (R_{\mu\nu} - 1/2\ g_{\mu\nu}R) \qquad (58)$$

The same result can be obtained from another variational principle known as the <u>Palatini variational principle</u>. We retain Eq.(54a) for the action, but now we regard the 10 $g_{\mu\nu}$'s and 40 $\Gamma^{\lambda}_{\mu\nu}$'s as independent variables. The usual relation between $\Gamma^{\lambda}_{\mu\nu}$ and $g_{\mu\nu}$ and its derivatives is not assumed to hold. As will be seen, it follows from the variational principle. The variation of the action is again given by Eq.(54b) with the $\delta\Gamma$'s contained in $\delta R_{\mu\nu}$ according to Eq.(56). Now

$$(\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} = \delta\Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\eta\nu}\delta\Gamma^{\eta}_{\mu\lambda} - \Gamma^{\eta}_{\mu\nu}\delta\Gamma^{\lambda}_{\eta\lambda} - \Gamma^{\eta}_{\lambda\nu}\delta\Gamma^{\lambda}_{\mu\eta} \qquad (60)$$

with a similar equation for the second term in Eq.(56). We integrate the terms containing derivatives by parts transfering the derivatives to $g^{\mu\nu}$ and giving surface terms that vanish. Then, rearranging terms we find

$$\delta S_g = (c^3/16\pi G)\int d^4x\ \sqrt{(-g)}\ ((R_{\mu\nu} - 1/2\ g_{\mu\nu}R)\delta g^{\mu\nu}$$
$$- g^{\mu\nu}{}_{;\nu}\delta\Gamma^{\lambda}_{\mu\lambda} + g^{\mu\nu}{}_{;\lambda}\delta\Gamma^{\lambda}_{\mu\nu}) \qquad (61)$$

We have used the usual notation for covariant derivatives of the metric tensor. We obtain Eq.(58) again and in addition the 40 functional derivatives

$$\delta S_g/\delta \Gamma^{\lambda}_{\mu\nu} = (c^3/16\pi G)\ \surd(-g)\ g^{\mu\nu}{}_{;\lambda} \tag{62a}$$

where

$$g^{\mu\nu}{}_{;\lambda} = \partial_\lambda g^{\mu\nu} + \Gamma^{\mu}_{\lambda\eta} g^{\eta\nu} + \Gamma^{\nu}_{\lambda\eta} g^{\mu\eta} \tag{62b}$$

Setting these functional derivatives equal to zero, we may solve to obtain the usual expression for the connection coefficients

$$\Gamma^{\lambda}_{\mu\nu} = 1/2\ g^{\lambda\sigma}[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}] \tag{63}$$

Setting Eq.(58) equal to zero we obtain the gravitational field equations for the vacuum; that is, Eq.(22) with the right hand side equal to zero. To obtain the right hand side from a variational principle, we must add to the gravitational action an action for matter and other fields which we shall denote by $S_m$. We write

$$S_m = 1/c \int d^4x\ \surd(-g)\ \Lambda(\phi^i,\ \partial\phi^i,\ g^{\mu\nu},\ \partial g^{\mu\nu}) \tag{64}$$

In general the integrand will depend not only on the fields $\phi^i$ and their derivatives but also on the metric tensor and its derivatives. Varying only the metric tensor and doing the usual integration by parts, we obtain

$$\begin{aligned} \delta S_m &= 1/c \int d^4x\ (\partial(\surd(-g)\Lambda/\partial g^{\mu\nu} - \partial_\lambda(\surd(-g)\Lambda/\partial\,\partial_\lambda g^{\mu\nu}))\delta g^{\mu\nu} \\ &\equiv 1/2c \int d^4x\ \surd(-g)\ T_{\mu\nu}\ \delta g^{\mu\nu} \end{aligned} \tag{65}$$

where we have defined the energy-momentum tensor $T_{\mu\nu}$ by

$$\begin{aligned} T_{\mu\nu} &= 2c/\surd(-g)\ \delta S_m/\delta g^{\mu\nu} \\ &= 2/\surd(-g)\ [\partial(\surd(-g)\Lambda/\partial g^{\mu\nu} - \partial_\lambda(\partial\surd(-g)\Lambda/\partial\,\partial_\lambda g^{\mu\nu})] \end{aligned} \tag{66}$$

We may now write $S = S_g + S_m$ and combine Eqs.(58) and (66) to obtain

$$\delta S/\delta g^{\mu\nu} = (c^3\surd(-g)/16\pi G)\ [(R_{\mu\nu} - 1/2\ g_{\mu\nu}R) + (8\pi G/c^4)\ T_{\mu\nu}] \quad (67)$$

Equating this to zero gives the gravitational field equations, Eq.(22)

---

Problem 2.

For a single scalar field $\Phi$ take

$$\Lambda = 1/2\ g^{\mu\nu}\ \partial_\mu \Phi\, \partial_\nu \Phi \ - V(\Phi)$$

(a) Show that the Euler-Lagrange equations are

$$\Phi^{;\lambda}{}_{;\lambda} - \partial V/\partial \Phi = 0$$

(b) Show that the energy-momentum tensor is

$$T_{\mu\nu} = \partial_\mu \Phi\, \partial_\nu \Phi + V(\Phi) \quad \times \quad (\partial_\mu\phi\, \partial_\nu\phi - \tfrac{1}{2} g_{\mu\nu} \partial_\alpha\phi\, \partial^\alpha\phi) + V(\phi)\, g_{\mu\nu}$$

Problem 3

For the electromagnetic field

$$\Lambda = -\ (1/16\pi)\ F_{\rho\sigma} F_{\alpha\beta} g^{\alpha\rho} g^{\beta\sigma}$$

(see Eq.(79) of Chap 8). (a) Use Eq.(66) to show that the energy-momentum tensor is

$$T_{\mu\nu} = (1/4\pi)\ [F_\mu{}^\sigma F_{\sigma\nu} + 1/4\ F_{\alpha\beta}F^{\alpha\beta}]$$

(b). Rewrite $\Lambda$ as

$$\Lambda = -1/8\pi\ (F^{\alpha\beta}(A_{\alpha,\beta} - A_{\beta,\alpha}) - 1/2\ F_{\alpha\beta}F^{\alpha\beta})$$

and treat $F^{\alpha\beta}$ and $A_\alpha$ as independent variables in a variational principle of the Palatini type. Show that

$$\delta S/\delta A_\alpha = (\sqrt{(-g)}/4\pi)\ F^{\alpha\beta}{}_{;\beta}$$

$$\delta S/\delta F^{\alpha\beta} = (\sqrt{(-g)}/8\pi\ [F_{\alpha\beta} - (A_{\alpha,\beta} - A_{\beta,\alpha})]$$

---

We now add the gravitational action to the action of Eq.(79) of Chap. 8 to obtain

$$S(a^\mu,\ A_\alpha,\ g^{\mu\nu}) = -\sum_a m_a c^2 \int da\ - \sum_a e_a/c \int A_\mu \dot{a}^\mu da$$
$$- (1/16\pi c) \int d^4x\ \sqrt{(-g)}\ F_{\mu\nu}F^{\mu\nu}$$
$$+ (c^3/16\pi G) \int d^4x\ \sqrt{(-g)}\ R$$
$$= S_p + S_{pf} + S_f + S_g \qquad (68)$$

This is the total action for a collection of charged particles, an electromagnetic field and a gravitational field. It was shown in Chap.8 that $\delta S/\delta a^\mu = 0$ gives the particle equations of motion, Eq.(1) of this Chapter, and that $\delta S/\delta A_\alpha$ gives the inhomogenous Maxwell's equations, Eq.(2a). The homogenous Maxwell's equations Eq.(2b), follow from the definition of $F_{\alpha\beta}$ in terms of $A_\alpha$. Finally, $\delta S/\delta g^{\mu\nu} = 0$ gives the gravitational field equations.

$$G_{\mu\nu} = R_{\mu\nu} - 1/2\ g_{\mu\nu}R = -(8\pi G/c^4)(T_{\mu\nu}^{(f)} + T_{\mu\nu}^{(p)}) \qquad (69)$$

The left side of the equation came from Eq.(58). The energy-momentum tensor for the electromagnetic field came from $\delta S_f/\delta g^{\mu\nu}$ according to Problem 3. The energy-momentum tensor for the particles comes from $\delta S_p/\delta g^{\mu\nu}$ mn via the calculation that follows. Write

$$S_p = -\sum_a m_a c^2 \int (g^{\mu\nu} da_\mu da_\nu)^{1/2} \qquad (70a)$$

$$\delta S_p = -\sum_a m_a c^2/2 \int da\, \dot{a}_\mu \dot{a}_\nu \delta g^{\mu\nu} \qquad (70b)$$

where we have used

$$(g^{\mu\nu} da_\mu da_\nu)^{-1/2} da_\mu da_\nu = \dot{a}_\mu \dot{a}_\nu da \qquad (70c)$$

We may put this into the form of Eq.(65) by multiplying by a 4-dimensional Dirac delta-function and integrating over spacetime to obtain

$$\delta S_p = - \sum_a m_a c^2/2 \iint d^4x da \; \delta^4(x^\alpha, a^\alpha) \; \dot{a}_\mu \dot{a}_\nu \; \delta g^{\mu\nu} \qquad (71)$$

Comparing this with Eq.(65) we identify

$$T_{\mu\nu}^{(p)} = - \sum_a m_a c^3 \int da \; \delta^4(x^\alpha, a^\alpha) \; \dot{a}_\mu \dot{a}_\nu \; (-g)^{-1/2} \qquad (72)$$

This variational principle gives in the very concise form, $\delta S = 0$, the laws of mechanics, electrodynamics and gravitation.

CONSERVATION OF ENERGY-MOMENTUM

Let us consider the action of Eq.(68) with the gravitational action omitted; that is

$$S = S_p + S_{pf} + S_f \qquad (73)$$

We write its variation as

$$\delta S = \int d^4x \ (\delta S/\delta a^\mu \ \delta a^\mu + \delta S/\delta A_\mu \ \delta A_\mu + \delta S/\delta g^{\mu\nu} \ \delta g^{\mu\nu}) \qquad (74)$$

Requiring the functional derivatives with respect to $a^\mu$ and $A_\mu$ vanish gives the equations of motion of particles and Maxwell's equations as has been shown. We are left with

$$\delta S = 1/2c \int d^4x \ \sqrt{(-g)} \ T_{\mu\nu} \ \delta g^{\mu\nu} \qquad (75)$$

where Eq.(66) has been used. We cannot argue that $\delta S$ should vanish for arbitrary variations of $g^{\mu\nu}$, for the gravitational action has been omitted. However, $\delta S$ should vanish for variations of $g^{\mu\nu}$ that are due to coordinate transformations, for S should be invariant to the choice of the coordinate system. Therefore, let us consider the infinitesimal coordinate transformation

$$x^\mu \rightarrow x^{\mu\prime} = x^\mu + \epsilon\xi^\mu(x) \qquad (76)$$

where $\epsilon$ is an infinitesimal constant and $\xi^\mu(x)$ is arbitrary. The transformation formula for $g^{\mu\nu}$ is

$$g^{\mu'\nu'}(x') = \partial_\alpha x^{\mu'} \partial_\beta x^{\nu'} g^{\alpha\beta}(x)$$

$$= g^{\mu\nu}(x) + \epsilon g^{\alpha\nu}\partial_\alpha\xi^\mu + \epsilon g^{\mu\beta}\partial_\beta\xi^\nu \qquad (77)$$

Also

$$g^{\mu'\nu'}(x') = g^{\mu'\nu'}(x + \epsilon\xi) = g^{\mu'\nu'}(x) + \epsilon\xi^{\lambda}\partial_{\lambda} g^{\mu\nu} \quad (78)$$

We define the variation of the metric tensor as

$$\delta g^{\mu\nu} = g^{\mu'\nu'}(x) - g^{\mu\nu}(x) = -\epsilon(\xi^{\lambda}\partial_{\lambda} g^{\mu\nu} - g^{\lambda\nu}\partial_{\lambda}\xi^{\mu} - g^{\mu\lambda}\partial_{\lambda}\xi^{\nu})$$

$$= -\epsilon\,[\mathcal{L}_{\xi}(g)]^{\mu\nu} \quad (79)$$

In the last step we have recognized the Lie derivative of the metric tensor. Note that we have taken the difference of $g^{\mu'\nu'}$ and $g^{\mu\nu}$ at the same space time point. The arguments of all of the factors in the integrand of Eq.(75) are variables of integration and must be the same. We substitute Eq.(79) into (75) and integrate by parts the two terms containg derivatives of $\xi$. We find

$$\delta S = -\epsilon/c \int d^4x\, \sqrt{(-g)}\,[1/\sqrt{(-g)}\,\partial_{\lambda}(\sqrt{(-g)}\,T_{\mu\nu} g^{\nu\lambda}) + 1/2\, T_{\alpha\beta}\partial_{\mu} g^{\alpha\beta}]\xi^{\mu}$$

$$= -\epsilon/c \int d^4x\, \sqrt{(-g)}\, T_{\mu}{}^{\lambda}{}_{;\lambda}\, \xi^{\mu} \quad (80)$$

We see that the requirement that S be invariant under infinitesimal coordinate transformations (that is, $\delta S = 0$ for arbitrary $\xi$) leads to

$$T_{\mu}{}^{\lambda}{}_{;\lambda} = 0 \quad (81)$$

which we recognize as the conservation of energy-momentum equation first encountered in Chap.8, Eq.(95b). Although we have done this for a system of particles and the electromagnetic field, it is clear that we could have included the action for other fields in S. The equations of motion and the field equations would cause all terms to drop out except for Eq.(75), and then Eq.(81) would follow. We have arrived at a very general result: <u>the conservation of energy momentum follows from the requirement of the invariance of the action under general coordinate transformations</u>.

We may use the same argument for the gravitational action whose variation we found to be

$$\delta S = (c^3/16\pi G) \int d^4x \, \sqrt{(-g)} \, (R_{\mu\nu} - 1/2 \, g_{\mu\nu} R) \, \delta g^{\mu\nu} \quad (82)$$

A repetition of the calculation that led to Eq.(81) gives

$$(R_{\mu\nu} - 1/2 \, g_{\mu\nu} R)_{;\nu} = 0 \quad (83)$$

which are the contracted Bianchi identities of Eq.(20). Thus, these identities are a consequence of the invariance of the gravitational action under general coordinate transformations.

## Chapter 11

## SOLUTIONS OF THE GRAVITATIONAL FIELD EQUATIONS

The gravitational field equations

$$G_{\mu\nu} = R_{\mu\nu} - 1/2\, g_{\mu\nu}R = -(8\pi G/c^4)T_{\mu\nu} \qquad (1)$$

are a set of ten coupled nonlinear partial differential equations. Only a small number of exact solutions are known. Generally, in looking for solutions one assumes a special form for the metric tensor, dictated by the symmetry of the physical problem. This form will contain one or more unknown functions. Then Eq.(1) reduces to a set of differential equations for these unknown functions.

A considerable amount of labor is involved just in writing down the differential equations once the form of the metric tensor has been chosen. One must calculate the 40 $\Gamma^\lambda{}_{\mu\nu}$ in terms of $g_{\mu\nu}$ and its derivatives. Then these are used in the calculation of $R_{\mu\nu}$ and R. It is worthwhile to employ all tricks that simplify this calculation. In Chapter 6 we introduced some of these tricks and used them to calculate curvature and Einstein tensors of interest. In this chapter we will use another set of tricks.

The Ricci tensor is obtained by contracting the curvature tensor; thus

$$\begin{aligned} R_{\beta\nu} &= R^\alpha{}_{\beta\alpha\nu} \\ &= \partial_\nu \Gamma^\alpha_{\beta\alpha} - \partial_\alpha \Gamma^\alpha_{\beta\nu} + \Gamma^\eta_{\beta\alpha}\Gamma^\alpha_{\nu\eta} - \Gamma^\eta_{\beta\nu}\Gamma^\alpha_{\alpha\eta} \end{aligned} \qquad (2a)$$

We may use

$$\Gamma^\alpha_{\beta\alpha} = (1/\surd(-g))\,\partial_\beta \surd(-g) \qquad (2b)$$

to write this in the form

$$R_{\beta\nu} = \partial_\beta \partial_\nu \, \mathrm{Ln}\sqrt{(-g)} - (1/\sqrt{(-g)})\, \partial_\alpha (\sqrt{(-g)}\, \Gamma^{\alpha}_{\beta\nu})$$

$$+ \Gamma^{\eta}_{\beta\alpha}\Gamma^{\alpha}_{\nu\eta} \qquad (2c)$$

The last term may be written as

$$\Gamma^{\eta}_{\beta\alpha}\Gamma^{\alpha}_{\nu\eta} = \mathrm{Tr}\,\Gamma_\beta \Gamma_\nu \qquad (2d)$$

where $\Gamma_\beta$ is the matrix with elements $\Gamma^{\eta}_{\beta\alpha}$. Note that as usual the superscript denotes the row and the subscript denotes the column. The first subscript labels the matrix.

The labor involved in calculating the connection coefficient can be reduced by writing the Lagrangian for a free particle of unit mass as

$$L = 1/2 \; g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \qquad (3a)$$

Then, comparing the equations of motion

$$d/d\tau \, (\partial L/\partial \dot{x}^{\mu}) - \partial L/\partial x^{\mu} = 0 \qquad (3b)$$

with

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = 0 \qquad (3c)$$

one easily finds $\Gamma^{\alpha}_{\mu\nu}$. These methods will be used in the following sections.

STATIC CENTRALLY SYMMETRIC SOLUTIONS

By static and centrally symmetric we mean that the metric must be independent of time and depend only on the rotational invariants $|d\vec{x}|^2$, $|\vec{x}|^2$ and $\vec{x}\cdot d\vec{x}$. Therefore, it must have the form

$$d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$$

$$= F(r)dt^2 - 2E(r)dt\vec{x}\cdot d\vec{x} - D(r)|\vec{x}\cdot d\vec{x}|^2 - C(r)|d\vec{x}|^2$$

$$= F(r)dt^2 - 2E(r)dtdr - D(r)r^2dr^2$$

$$- C(r)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \qquad (4)$$

We may eliminate the term containing dtdr by defining a new time coordinate by

$$t' = t + f(r) \qquad (5)$$

and choosing $Ff' = rE(r)$. We find the metric becomes

$$d\tau^2 = F(r)dt^2 - G(r)dr^2 - C(r)r^2(d\theta^2 + \sin^2\theta d\phi^2)] \qquad (6)$$

We have dropped the prime on $t'$ and defined a new function $G(r) = C(r) + r2D(r) + 2rE(r) - f'2(r)$. Next we define a new radial variable

$$r'^2 = C(r)r^2 \qquad (7)$$

and find that the metric takes the form

$$d\tau^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (8)$$

We have dropped the prime on $r'$ and collected the radial functions into the exponential functions $e^{\nu(r)}$ and $e^{\lambda(r)}$; their values in terms of the functions F, E, D and C is of no consequence.

The covariant components of the metric tensor are identified as

$$g_{\mu\nu} = \begin{vmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ & -e^{\lambda(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{vmatrix} \qquad (9a)$$

The contravariant components are

$$g^{\mu\nu} = \begin{vmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2}\sin^{-2}\theta \end{vmatrix} \qquad (9b)$$

The square root of the determinant is

$$\sqrt{(-g)} = r^2 \sin\theta e^{(\nu+\lambda)/2} \qquad (9c)$$

We now construct the Lagrangian, Eq.(3a), and write the equations of motion, Eq.(3b). As an example, for $\mu = 1$ we find

$$d/d\tau(-e^{\nu}\dot{r}) - 1/2\ \nu' e^{\nu}\dot{t}^2 + 1/2\ \lambda' e^{\lambda}\dot{r}^2$$

$$+ r(\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2) = 0 \qquad (10a)$$

The prime denotes a derivative with respect to r. This may be rewritten as

$$r + 1/2\ \lambda'\dot{r}^2 + 1/2\ e^{\nu-\lambda}\nu'\dot{t}^2 - e^{-\lambda}r\dot{\theta}^2$$

$$- e^{-\lambda}r\sin^2\theta\dot{\phi}^2 = 0 \qquad (10b)$$

Comparing this with Eq.(3c) for $\alpha = 1$, we identify

$$\Gamma^1_{00} = 1/2\ e^{\nu-\lambda}\nu', \qquad \Gamma^1_{11} = 1/2\ \lambda',$$

$$\Gamma^{\lambda}_{22} = -\ e^{-\lambda}r, \qquad \Gamma^1_{33} = -\ e^{-\lambda}r\sin^2\theta \tag{11}$$

The remaining $\Gamma$'s may be obtained from the other three equations of motion. These may be written as matrices as follows.

$$\Gamma_0 = \left| \begin{array}{cccc} 0 & \nu'/2 & 0 & 0 \\ \nu'/2\ e^{\nu-\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right|$$

$$\Gamma_1 = \left| \begin{array}{cccc} \nu'/2 & 0 & 0 & 0 \\ 0 & \lambda'/2 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1/r \end{array} \right|$$

$$\Gamma_2 = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{-\lambda}r & 0 \\ 0 & 1/r & 0 & 0 \\ 0 & 0 & 0 & \cot\theta \end{array} \right| \tag{12}$$

$$\Gamma_3 = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-\lambda}r\sin^2\theta \\ 0 & 0 & 0 & -\sin\theta\cos\theta \\ 0 & 1/r & \cot\theta & 0 \end{array} \right|$$

We now have all that is needed for the calculation of $R_{\beta\nu}$ using Eq.(2). Then $G^{\nu}_{\mu}$ may be calculated. The results are:

$$G^0_0 = e^{-\lambda}[1/r^2 - \lambda'/r] - 1/r^2 \tag{13a}$$

$$G^1_1 = e^{-\lambda}[1/r^2 + \nu'/r] - 1/r^2 \tag{13b}$$

$$G^2_2 = G^3_3 = e^{-\lambda}[\nu''/2 + \nu'^2/4 - \lambda'/2r + \nu'/2r - \lambda'\nu'/4] \tag{13c}$$

These results will now be applied.

(a) Schwarzschild Solution

We set the right hand side of Eq.(1) equal to zero and look for a vacuum solution. Subtracting Eq.(13a) from (13b) gives

$$e^{-\lambda}(\nu' + \lambda')/r = 0 \tag{14a}$$

from which

$$\nu + \lambda = \text{constant} \tag{14b}$$

By inspection of Eq.(8) it is seen that if a constant is added to $\nu$, the constant may be absorbed by a redefinition of t. There is no loss of generality by taking the constant in Eq.(14b) equal to zero, and so $\lambda = -\nu$. Defining $f(r) = e^{-\lambda}$, Eq.(13a) takes the form

$$df/dr + f/r = 1/r \tag{15a}$$

which has the solution

$$f = 1 + A/r = e^{-\lambda} \tag{15b}$$

where A is a constant of integration. It is easily checked that Eq.(13c) is equal to zero. It will be convenient to define $A = -2GM/c^2 = -r_s$ and write Eq.(8) as

$$d\tau^2 = (1 - r_s/r)dt^2 - (1 - r_s/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{16}$$

This is known as the Schwarzschild metric; $r_s$ is called the Schwarzschild radius. As will be seen in the next section, M may be interpreted as the mass of a particle located at $r = 0$ that is the source of the gravitational field described by Eq.(16).

## (b) Reissner-Nordstrom Solution

We may modify the Schwarzschild solution by assuming that the particle at the origin carries an electric charge q. Then the non-vanishing elements of the electromagnetic energy-momentum tensor are

$$T_0^0 = T_1^1 = - T_2^2 = - T_3^3 = E^2/8\pi = q^2/8\pi r^4 \qquad (17)$$

Two components of Eq.(1) are

$$G_0^0 = e^{-\lambda}[1/r^2 - \lambda'/r] - 1/r^2 = - Gq^2/c^4r^4 \qquad (18a)$$

$$G_1^1 = e^{-\lambda}[1/r^2 + \nu'/r] - 1/r^2 = - Gq^2/c^4r^4 \qquad (18b)$$

Once again we find $\lambda = -\nu$. The solution is now easily found to be

$$e^{-\lambda} = e^{\nu} = 1 - r_s/r + Gq^2/c^4r^2 \qquad (19)$$

The metric

$$d\tau^2 = (1 - r_s/r + Gq^2/c^4r^2)dt^2 - (1 - r_s/r + Gq^2/c^4r^2)^{-1}dr^2$$

$$- r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (20)$$

is known as the Reissner-Nordstrom metric.

## (c) Interior of a Star

For the energy-momentum tensor we shall take

$$T_{\mu\nu} = (\rho + \rho U/c^2 + P/c^2)u_\mu u_\nu - g_{\mu\nu}P \qquad (21a)$$

with

$$u_r = u_\theta = u_\phi = 0 \qquad (21b)$$

This describes a fluid at rest. Since

$$g^{\mu\nu}u_\mu u_\nu = c^2 \tag{21c}$$

we must take $u_t = ce^{\nu/2}$. The nonvanishing components of the energy-momentum tensor are:

$$T_0^0 = \rho c^2 + \rho U \equiv \epsilon$$

$$T_1^1 = T_2^2 = T_3^3 = -P \tag{21d}$$

Two components of Eq.(1) are

$$e^{-\lambda}[1/r^2 - \lambda'/r] - 1/r^2 = -8\pi G\epsilon/c^4 \tag{22a}$$

$$e^{-\lambda}[1/r^2 + \nu'/r] - 1/r^2 = +8\pi GP/c^4 \tag{22b}$$

Eq.(22a) can be written as

$$d/dr(re^{-\lambda}) = 1 - (8\pi G/c^4)\ \epsilon(r)r^2 \tag{23a}$$

This can be integrated to obtain

$$e^{-\lambda} = 1 - 2GM(r)/c^2r \tag{23b}$$

where

$$M(r) = 4\pi/c^2\int_0^r r^2dr\ \epsilon(r) \tag{23c}$$

may be interpreted as the mass contained within the radius r. Using this in Eq.(22b)we obtain

$$r^2\nu' = e^{\lambda}[M(r) + 4\pi r^3P/c^2] \tag{24}$$

The divergence of the energy-momentum tensor must vanish. Thus $T^{\nu}_{\mu;\nu} = 0$. For $\mu = 1$ this gives

$$dP/dr + 1/2\,(P + \epsilon)\nu' = 0 \qquad (25)$$

This may be used to eliminate $\nu'$ from Eq.(24) and the results written as

$$-dP/dr = GM(r)\epsilon/c^2r^2[1+P/\epsilon][1+4\pi r^3P/c^2M(r)][1-2GM(r)c^2r]^{-1} \qquad (26)$$

This equation is known as the Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium. The three factors in square brackets are relativistic corrections to the Newtonian hydrostatic equilibrium equation

$$-dP/dr = GM(r)\rho/r^2 \qquad (27)$$

The physical interpretation is that outward force due to the pressure gradient is balanced by the gravitational attraction of the mass contained within r.

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Problem 1.

Consider the following (unrealistic) model of a star. The internal energy density U is zero. The mass density $\rho$ is constant for $r \leq R$ and zero for $r > R$. Integrate Eq.(27) and show that the pressure is given by

$$P(r) = \rho c^2 \left[\frac{(1 - 2M/c^2R)^{1/2} - (1 - 2GMr^2/c^2R^3)^{1/2}}{(1 - 2GMr^2/c^2R^3)^{1/2} - 3(1 - 2GM/c^2R)^{1/2}}\right] \qquad (28)$$

where M is the total mass of the star. Show that no solution with finite pressure can exist unless $R > 2GM/c^2$.

## EXPERIMENTAL TEST OF THE THEORY

Newton's theory of gravitation is remarkably accurate for phenomena on the earth and in the solar system. Consequently, the experimental tests of general relativity have involved small effects which until recent years have been barely observable. As a result general relativity has been a somewhat uninteresting field for the experimentalist, but there are indications that this is now changing due to recently introduced techniques.

The three classical tests of general relativity are (a) motion of the perihelia of planetary orbits, (b) the bending of light rays in a gravitational field, and (c) the gravitational red shift. All three tests were suggested in Einstein's 1916 paper.

### (a) Motion of the perihelia of planetary orbits

We use Eqs.(3a) and (16) to write the Lagrangian for a particle of unit mass in a Schwarzchild metric as

$$L = 1/2\ g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

$$= 1/2\ (e^{\nu}\dot{t}^2 - e^{-\nu}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2) \qquad (28a)$$

where

$$e^{\nu} = (1 - r_s/r) \qquad (28b)$$

Since the orbit of a particle in a gravitational field is independent of its mass, it is sufficient to consider a particle of unit mass. The motion is confined to a plane which we will take to be the equatorial plane $\theta = \pi/2$. The Lagrangian is independent of t and $\phi$, so the momenta

$$p_t = e^{\nu}\dot{t} \qquad (29a)$$

and

$$p_\phi = -r^2\dot{\phi} \qquad (29b)$$

are constant

Since the Lagrangian is independent of the proper time $\tau$, the Hamiltonian

$$H = L = 1/2\ (d\tau/d\tau)^2 = 1/2 \qquad (29c)$$

is a constant of the motion.

Using Eqs.(29a) and (29b) in (29c) gives the first order differential equation for r

$$e^{-\nu}(p_t^{\ 2} - \dot{r}^2) - p_\phi^{\ 2}/r^2 = 1 \qquad (30a)$$

Using Eq.(29b) to change the independent variable from $\tau$ to $\phi$ and changing the dependent variable from r to u = 1/r, we obtain

$$(du/d\phi)^2 + u^2(1 - r_s u) = (p_t^{\ 2} - 1)/p_\phi^{\ 2} + 2GMu/p_\phi^{\ 2} \qquad (30b)$$

We have used $r_s$ = 2GM in the last term. Differentiating gives

$$d^2u/d\phi^2 + u - (3r_s/2)u^2 = GM/p_\phi^{\ 2} \qquad (30c)$$

The term containing $r_s$ is a relativistic correction. If this term is neglected, the equation may be solved with the result

$$u = 1/r = \frac{1 + e\cos\phi}{a(1 - e^2)} \qquad (31a)$$

where

$$a(1 - e^2) = p_\phi^{\ 2}/GM \qquad (31b)$$

This is the equation of an ellipse of major radius a and eccentricity e. We shall now solve Eq.(30c) assuming that the orbit is almost circular and treating the term containing $r_s$ as a small correction. For a circular orbit

$$u - (3r_s/2)u^2 = GM/p_\phi{}^2 \tag{32a}$$

Call the solution of this $u_0$ and define it to be

$$u_0 = 1/a(1 - e^2) \tag{32b}$$

Now write

$$u(\phi) = u_0 + u_1(\phi) \tag{32c}$$

and assume $u_1 \ll u_0$. Neglecting a term in $u_1{}^2$, we find that $u_1$ satisfies

$$d^2u_1/d\phi^2 + (1 - 3r_s u_0)u_1 = 0 \tag{32d}$$

which has the solution

$$u_1 = A\cos\omega\phi \tag{32e}$$

where

$$\omega = (1 - 3r_s u_0)^{1/2} \tag{32f}$$

Setting the constant of integration A equal to $eu_0$ we obtain

$$u = \frac{1 + e\cos\omega\phi}{a(1 - e^2)} \tag{32g}$$

This is the equation for a precessing ellipse. The minimum value of r occurs at $\phi = 0$ and again at $\phi = 2\pi/\omega$. Therefore the precession per orbit is

$$\delta\phi = 2\pi(1/\omega - 1) \simeq 6GM/a(1 - e^2) \qquad (33)$$

This may be divided by the period of an orbit

$$T = 2\pi a^{3/2}/(GM)^{1/2} \qquad (34a)$$

to obtain

$$\delta\phi/T = 6GM/Ta(1 - e^2) \qquad (34b)$$

for the rate of precession.

When the numbers are put into Eq.(34), one finds an advance of the perihelion of 42.9" per century for the planet Mercury, 8.6" for Venus, 3.8" for Earth, and 1.35" for Mars.

This precession is most easily observed for Mercury because of the large eccentricity of its orbit. There is a precession due to the perturbation of Mercury's orbit by the other planets. The principal contributions are 278" per century due to venus, 90" due to earth, and 154" due to Jupiter. When all of these contributions are taken into account, there remains a discrepancy of 43" per century. This discrepancy was known to astronomers before Einstein's paper was published and had been the subject of some speculation. It was thought that it might indicate the presence of an unobserved planet between Mercury and the sun. In anticipation of its discovery this hypothetical planet was named Vulcan, but despite careful searching of the skies at the time of solar eclipses no planet could be found. Einstein's theory provided almost exactly the right precession to account for the discrepancy. This was one of the most dramatic early successes of the theory. More recently, Hulse and Taylor discovered a pulsar in a binary system. This system consisted of two compact bodies in an elliptical orbit with a maximum separation of only about $a = 10^{11}$ cm. (about the radius of the sun). This small value of a makes the rate of precession about 4 degrees per year. A pulsar emits pulses of radio waves in a regular, clock-like fashion whose arrival time at the earth can be measured with high precission. This makes possible an accurate determination of the orbital parameters of the binary system. The measured precession agrees well with the predictions of Einstein's theory. Futhermore, this system should lose energy by the radiation of gravitational waves causing a decrease of the orbital radius and an increase in orbital frequency. This has been observed, and again the agreement with Einstein's theory is good.

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Problem 2.

Rewrite Eq.(30a) as

$$1/2\ \dot{r}^2 + V(r) = (p_t^{\ 2} - 1)/2 = E$$

where

$$V = p_\phi^{\ 2}/2r^2 - r_s/2r - p_\phi^{\ 2} r_s/2r^3$$

is an effective potential. Show that all orbits with $p_\phi < \sqrt{3}\ r_s$ arrive at $r = 0$.

Problem 3.

Consider radial motion of particles in a Schwarzschild metric. Replace the constant of the motion $p_t$ by $r_m$ by defining

$$p_t^{\ 2} - 1 = -\ k r_s/r_m$$

where $k = \pm 1,\ 0$. Change the independent variable from $\tau$ to $\eta$ by defining

$$d\tau = \sqrt{\frac{r_m}{r_s}}\, r\, d\eta$$

Show that the solutions are

$$r = r_m/2k\,(1 - \cos\eta\sqrt{k})$$

$$\tau - \tau_0 = 1/2k\,(r_m{}^3/r_s)^{1/2}\,(\eta - 1/\sqrt{k}\,\sin\eta\sqrt{k})$$

$$t - t_0 = (r_m/2k)(r_m/r_s - k)^{1/2}(\eta - 1/\sqrt{k}\,\sin\eta\sqrt{k})$$

$$+ (r_s r_m - k r_s{}^2)^{1/2}\,\eta$$

$$+ r_s \ln \left| \frac{\tan\eta\sqrt{k}/2 - (r_m/kr_s - 1)^{-1/2}}{\tan\eta\sqrt{k}/2 + (r_m/kr_s - 1)^{-1/2}} \right|$$

These results will be used in a later section.

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(b) The Bending of Light Rays and Planetary Radar Reflections

A photon moves along a spacetime path with $d\tau = 0$. We may use our previous derivation of the orbit until Eq.(~~31c~~ 34c) is obtained and at this point set $d\tau = 0$. From Eq.(29b) we see that $d\tau = 0$ implies $p_\phi = \infty$. This gives

$$d^2u/d\phi^2 + u - 3r_s u^2/2 = 0 \qquad (35)$$

as the equation for the orbit of a photon. The effects of the gravitational field are contained in the third term. If this term is neglected a solution is found to be

$$u = \frac{\cos\phi}{R} = 1/r \qquad (36a)$$

or

$$R = r\cos\phi \qquad (36b)$$

This is the equation of a straight line that passes the origin (where the gravitating body is located) at a distance of R. We shall call R the impact parameter. We shall solve Eq.(35) approximately treating the third term as a perturbation. We write $u = u_0 + u_1$ where $u_0$ is given by Eq.(36). Then, neglecting terms in $u_1^2$ and $r_s u_1$ we obtain the equation

$$d^2u_1/d\phi^2 + u_1 = (3r_s/2R^2)\cos^2\phi \qquad (37)$$

We add the particular integral of this equation to $u_0$ to obtain

$$u = \frac{\cos\phi}{R} - \frac{3r_s}{4R^2}(1 - 1/3\cos 2\phi) \qquad (38)$$

In terms of the Cartesian coordinated $x = r\cos\phi$ and $y = r\sin\phi$, this equation may be written as

$$R = x + \frac{r_s}{2R}\left[\frac{x^2 + 2y^2}{(x^2 + y^2)^{1/2}}\right] \qquad (39)$$

This gives the path in the x-y plane sketched in Fig.1

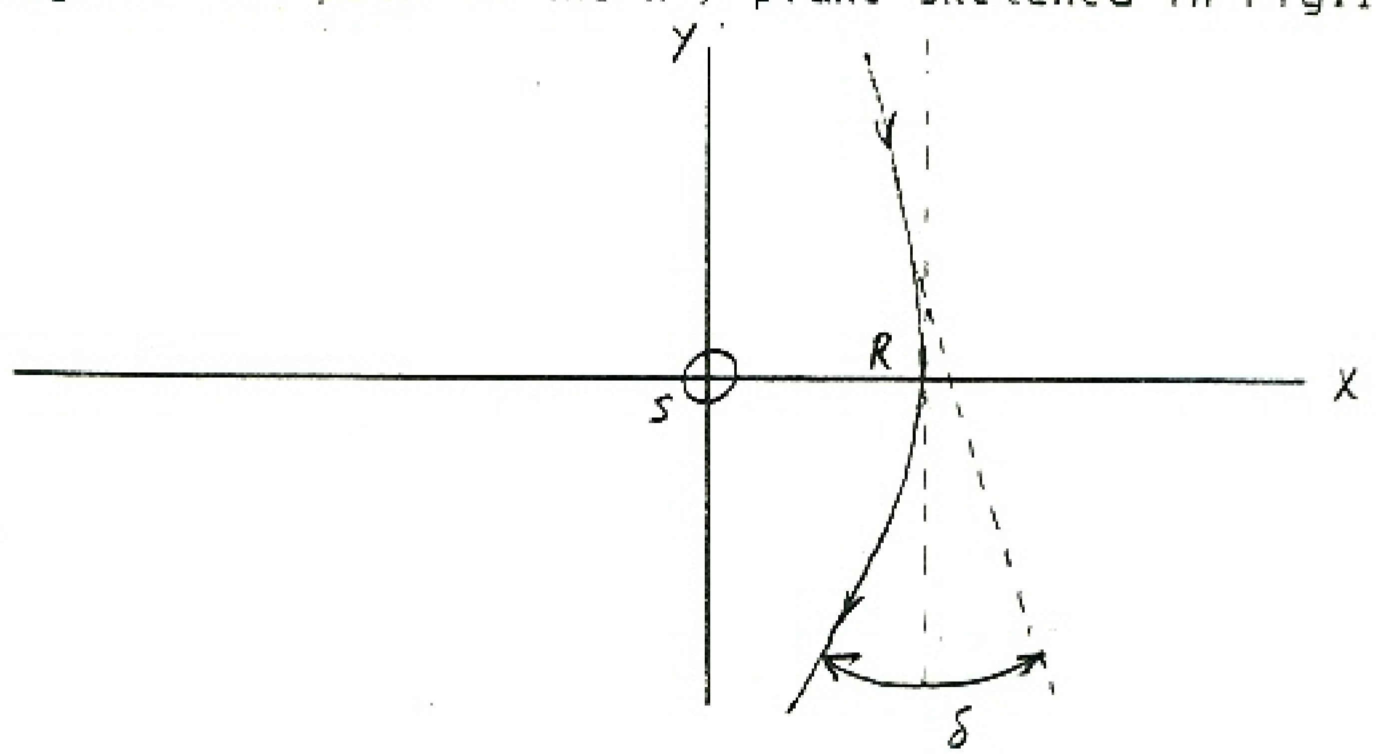

Fig.1

For very large |y| we obtain

$$R = x \pm yr_s/R \quad (40a)$$

and

$$dx/dy = \pm r_s/R \quad (40b)$$

The angular deflection of the photons path by the gravitational field is

$$\delta = 2r_s/R \quad (40c)$$

For a ray of light from a star just grazing the edge of the sun, $r_s$ = 3 km and R = 700,000 km, so $\delta$ is about 1.75".

In 1919, only three years after the publication of Einstein's paper, a particularly favorable eclipse of the sun made it possible to photograph starlight which passed very close to the sun and to detect small apparent changes in the positions of stars. Expeditions were sent by the Royal Society and the Royal Astronomical Society of Great Britain to favorable positions in Africa and South America for viewing the eclipse. The measured displacements of the positions of stars near the solar disc agreed very well with the calculated values.

Recently another method has been devised for measuring the changes in the propagation characteristics of light near a massive body such as the sun, This method involves reflecting a radar beam from a planet and measuring the time required for a photon to make the trip from the earth to the planet and back. Consider Fig.2. Naïvely, one would expect the travel time to be $t = 2(a + b)/c$, but because the photon is moving near the sun small corrections are required.

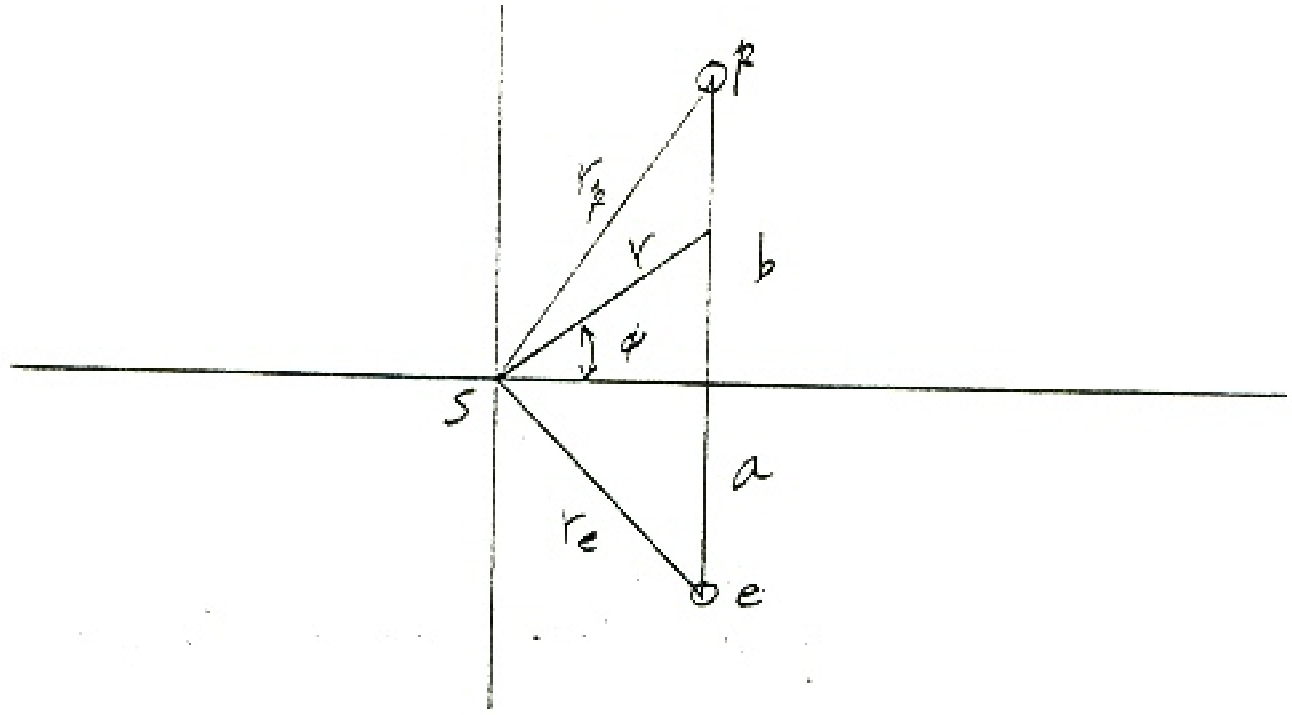


Fig.2

The path of a photon is given by $d\tau = 0$. Using the Schwarzschild metric with $\theta = \pi/2$ this is

$$d\tau^2 = 0 = (1 - r_s/r)dt^2 - \frac{dr^2}{c^2(1 - r_s/r)} - \frac{r^2 d\phi^2}{c^2} \qquad (41)$$

From inspection of Fig.1 we find

$$r = \frac{R}{\cos\phi}, \qquad dr = \frac{R\sin\phi\, d\phi}{\cos^2\phi} \qquad (42)$$

These may be used to eliminate r and dr from Eq.(41) which then becomes

$$dt2 = \frac{R^2 d\phi 2}{c^2\cos^2\phi}\left[\frac{1}{(1 - r_s/R\cos\phi)} + \frac{\sin^2\phi}{(1 - r_s/R\cos\phi)^2\cos^2\phi}\right] \qquad (43)$$

Since $r_s/R$ is a very small quantity, we can neglect all but the first-order terms and obtain

$$dt = \frac{Rd\phi}{c}\left[\frac{1}{\cos^2\phi} + \frac{r_s(1 + \sin^2\phi)}{2R\cos\phi}\right] \qquad (44)$$

Integrating between the limits shown in the figure and multiplying by 2, we find that the round trip time is

$$t = \frac{2(a + b)}{c} + \frac{r_s}{c}\left[2\log\frac{(r_p + b)(r_e + a)}{R^2} - \left(\frac{b}{r_p} + \frac{a}{r_e}\right)\right] \qquad (45)$$

The first term is the result that would be obtained using Euclidean geometry. The second term is the relativistic correction.

In a typical measurement the first term is of the order of 1000 sec, while the second term is of order 100 μsec. Present-day techniques are good enough to allow measurements to about 10 μsec. Thus the experiments are capable of checking the predictions of general relativity to about 10% accuracy. There is good agreement between theory and experiment.

---

Problem 4.

Use Eq.(35) to show that a photon approaching a massive body with impact parameter R less than $3^{3/2}GM/c^2$ will be captured.

## (c) The Gravitational Red Shift

Factoring out $dt^2$ in the Schwarzschild metric gives

$$d\tau^2 = dt^2[(1 - r_s/r) - \frac{v_r^2}{(1 - r_s/r)} - v_\theta^2 - v_\phi^2] \qquad (46)$$

where $v_r$, $v_\theta$ and $v_\phi$ are the components of velocity in spherical coordinates. When $r_s = 0$ we obtain the relation between dt and dt

$$d\tau = dt(1 - v2)1/2 \qquad (47)$$

that is familiar from the special theory of relativity. For a particle at rest in a gravitational field, we find

$$d\tau = dt(1 - r_s/r)^{1/2} \qquad (48)$$

The period of oscillation of an atom at r is related to the period of oscillation of an identical atom at $r = \infty$ by

$$T_r = T_\infty(1 - r_s/r)^{1/2} \qquad (49)$$

The frequency shift is given by

$$\Delta\nu/\nu = - \Delta T/T = (T_r - T_\infty)/T_\infty \simeq - r_s/2r = GM/c^2r \qquad (50)$$

For an atom on the surface of the sun this is a very small shift of about $2 \times 10^{-6}$. In the case of the companion of Sirius, which is a very dense white dwarf star, the shift is about 30 times larger. These measurements are very difficult because the shift is so small, but they have been made and show agreement with the theory.

In 1960 Mossbauer discovered an effect that made it possible to observe the gravitational red shift in the laboratory. Under certain conditions some nuclei such as $^{57}Fe$ emit gamma rays of extremely sharply defined frequency. A crystal of the same element will resonantly absorb these gamma rays. Anything that shifts the frequency of these gamma rays, such as the gravitational red shift or the Doppler effect, will reduce the absorption. Consider two such crystals, a source and an absorber, separated by a vertical distance h in the earth's gravitational field. The frequency difference $\Delta\nu$ between sourve and absorber is given by

$$\Delta\nu/\nu = gh/c^2 \qquad (51)$$

where g is the gravitational acceleration at the earth's surface. For h = 10 meters this is only about $10^{-15}$. Although these is extremely small it appreciably reduces the absorbtion. The absorption can be restored by moving the absorber downward with a velocity v. This produces a frequency shift due to the Doppler effect given by

$$\Delta\nu/\nu = v/c \qquad (52)$$

The velocity necessary to compensate for the gravitational shift is only $v = gh/c = 3 \times 10^{-5}$ cm/sec. This experiment was done by Pound and Rebka in 1960. The gravitational red shift was measured to an experimental accuracy of about 1% and good agreement with theory was found.

THE KRUSKAL-SZEKERES EXTENSION OF THE SCHWARZSCHILD METRIC

Inspection of the Schwarzschold metric, Eq.(16) shows that one or another of the components of the metric tensor becomes infinite at $r = 0$ and $r = r_s$. These singularities may be due either to true singularities of the spacetime geometry or to the failure of the coordinate system chosen to properly cover a region of spacetime. It is not immediately obvious which is the case. If a singularity is due to a choice of coordinate system, then it should be possible to remove it by transforming to a new and better behaved coordinate system.

One might expect that the curvature would become infinite at a true spacetime singularity. However, the components of the curvature tensor $R_{\mu\gamma\alpha\beta}$ depend on the choice of coordinates, so again singularities in these components may be due to the choice of coordinate system. The curvature scalar R is independent of the choice of coordinates. Now, $R = 0$ follows from contraction of the Einstein vacuum equations $R_{\mu\gamma} - g_{\mu\gamma}/2\ R = 0$. We may conclude that the apparent singularity at $r = r_s$ is a coordinate singularity rather than a singularity in spacetime. We cannot draw such a conclusion about the point $r = 0$, since examination of R shows that it vanishes because of cancellation of terms that vary as inverse powers of r and so become infinite at $r = 0$. It is not mathematically legitimate to set $\infty - \infty = 0$. Indeed, our knowledge of Newtonian gravitation leads us to expect a singularity at the point $r = 0$ which is the location of the point masss that is the source of the gravitational field. These considerations motivate us to search for a new coordinate system in which the coordinate singularity at $r = r_s$ is removed.

Before considering the Schwarzschild metric it is useful to examine some simpler cases of coordinate singularities. First, consider the metric for three dimensional flat space

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\ d\phi^2) \qquad (53)$$

The contravariant component of the metric tensor $g^{\phi\phi}$ becomes infinite at $r = 0$ and at $\theta = 0$ and $\pi$. However, we know that the coordinate transformation

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi \qquad (54)$$

$$z = r\cos\theta$$

yields the metric

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad (55)$$

without singularities.

As a second example consider the metric

$$d\tau^2 = t^{-4}dt^2 - dx^2 - dy^2 - dz^2 \qquad (56a)$$

defined over the coordinate ranges $-\infty < x,y,z < +\infty$, $0 < t < \infty$. The metric appears to have a singularity at $t = 0$ and the coordinate range of $t$ was chosen to avoid crossing this singularity. The coordinate transformation

$$t' = 1/t \qquad (56b)$$

yields the metric

$$d\tau^2 = dt'^2 - dx^2 - dy^2 - dz^2 \qquad (56c)$$

without singularities. Futhermore we can extend the range of $t'$ to $-\infty < t' < \infty$. The original spacetime is seen to be the half of Minkowski space with $t' > 0$. The coordinate transformation, Eq.(56b), has removed the singularity and made it possible to extend the range of coordinates to cover all of Minkowski space.

In these two examples the coordinate transformations that removed the singularities were rather easily found. Our third example is somewhat more subtle and also more closely related to the Schwarzschild metric. Consider the metric

$$d\tau^2 = x^2 dt^2 - dx^2 - dy^2 - dz^2 \qquad (57a)$$

with the coordinate ranges $-\infty < t,y,z < +\infty$ , $0 < x < \infty$. The metric appears to have a singularity at $x = 0$ and the coordinate range was chosen to avoid crossing this singularity. Since the coordinates $y$ and $z$ will play no role in the work that follows, we shall for simplicity omit them and work with the 2-dimensional spacetime with metric

$$d\tau^2 = x^2 dt^2 - dx^2 \qquad (57b)$$

This spacetime is called Rindler spacetime.

If we were to calculate the scalar curvature for this metric, we would find that $R = 0$ and there was no singular behavior at $x = 0$. (We omit the calculation). We should be able to find a coordinate system in which the metric is the Minkowski metric, but it is not so obvious how to find the required coordinate transformation. A useful trick that works in the Schwarzschild case as well as here is to take as coordinate curves the null geodesics. That is we write the null condition

$$d\tau^2 = 0 = x^2 dt^2 - dx^2 \qquad (58a)$$

and solve

$$dt/dx = \pm 1/x \qquad (58b)$$

to obtain

$$t = \pm \ln x + \text{constant} \qquad (58c)$$

We now define null coordinates $(u, v)$ by

$$u = t - \ln x, \quad v = t + \ln x \qquad (59)$$

The coordinate $u$ is constant along an outgoing null geodesic and $v$ is constant along an incoming null geodesic. In these coordinates the metric is

$$d\tau^2 = e^{v-u} du dv \tag{60}$$

Now, an obvious change of coordinates

$$U = -e^{-u}, \qquad V = e^{v} \tag{61a}$$

gives an even simpler metric

$$d\tau^2 = dUdV \tag{61b}$$

The ranges of $U$ and $V$ that correspond to the ranges of $x$ and $t$ are easily found to be $-\infty < U < 0$, $0 < V < +\infty$. However, since Eq.(61b) is free of singularities we can extent these ranges to $-\infty < U < +\infty$, $-\infty < V < +\infty$. A further transformation

$$T = (V + U)/2, \qquad X = (V - U)/2 \tag{62a}$$

gives the Minkowski metric

$$d\tau^2 = dT^2 - dX^2 \tag{62b}$$

with the coordinate ranges $-\infty < X < +\infty$, $-\infty < T < +\infty$.

It is now a simple matter to find the transformations connecting the old coordinates $(t,x)$ and the new coordinates $(T,X)$; these are

$$x = (X^2 - T^2)^{1/2}, \qquad t = \tanh^{-1}(T/X) \tag{63a}$$

$$X = x\cosh t, \qquad T = x\sinh t \tag{63b}$$

Curves of constant $x$ and $t$ are sketched in Fig.3. It is seen that

Rindler spacetime is simply the wedge X > |T| of Minkowski spacetime. Once again, we have succeeded in finding a coordinate transformation that has removed an apparent singularity and allowed an extension of spacetime.

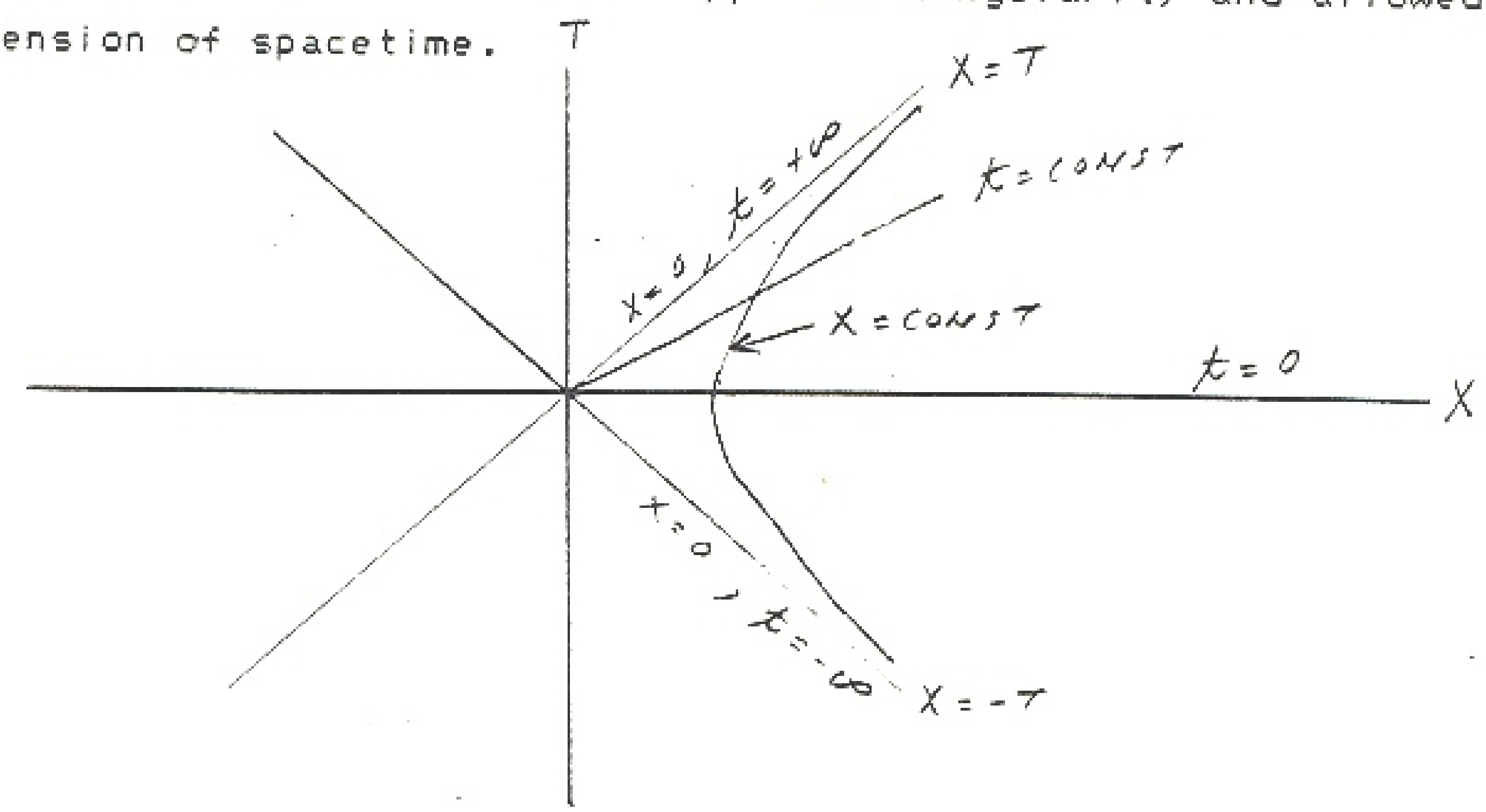


Fig.3'

Now we turn our attention to the Schwarzschild spacetime with metric given by Eq.(16). Since the coordinates θ and ϕ will not be changed by the transformations that follow, we shall ignore them and for simplicity consider the 2-dimensional spacetime with metric

$$d\tau^2 = (1 - r_s/r)dt^2 - \frac{dr^2}{(1 - r_s/r)} \qquad (64)$$

As in the case of Rindler spacetime, we shall take as coordinate curves null geosesics. We set $d\tau = 0$ and solve

$$dr/dt = \pm (1 - r_s/r) \qquad (65a)$$

to obtain

$$t = \pm r^* + \text{constant} \qquad (65b)$$

where

$$r^* = r + r_s \ln(r/r_s - 1) \tag{65c}$$

is called the <u>Regge-Wheeler tortoise coordinate</u>. Now we introduce new coordinates (u,v) by

$$u = t - r^*, \qquad v = t + r^* \tag{66a}$$

$$t = (v + u)/2, \quad r^* = (v - u)/2 \tag{66b}$$

In these new coordinates the metric is

$$d\tau^2 = (1 - r_s/r)dudv \tag{66c}$$

which may also be written as

$$d\tau^2 = (r_s/r)e^{-r/r_s} e^{(v - u)/2r_s} dudv \tag{66d}$$

Next we make the transformation of coordinates

$$U = - e^{-u/2r_s}, \qquad V = e^{v/2r_s} \tag{67a}$$

and obtain the metric

$$d\tau^2 = (4r_s^3/r)e^{-r/r_s} dUdV \tag{67b}$$

Finally, the change of coordinates given in Eq.(62a) gives the metric found independently by Kruskal and Szekeres in 1960,

$$d\tau^2 = (4r_s^3/r)e^{-r/r_s} (dT^2 - dX^2) - r^2(d\theta^2 - \sin^2\theta\, d\phi^2) \tag{68}$$

We have added on the terms in the coordinates $\theta$ and $\phi$ that were omitted at the beginning of this derivation. In Eq.(68) r is to be regarded as a function of the new coordinates X and T given implicitly by

$$(r/r_s - 1)e^{r/r_s} = X^2 - T^2 \qquad (69a)$$

and t is given by

$$t = 2r_s \tanh^{-1}(T/X) \qquad (69b)$$

Curves of constant r and t in the X-T plane are sketched in Fig.4

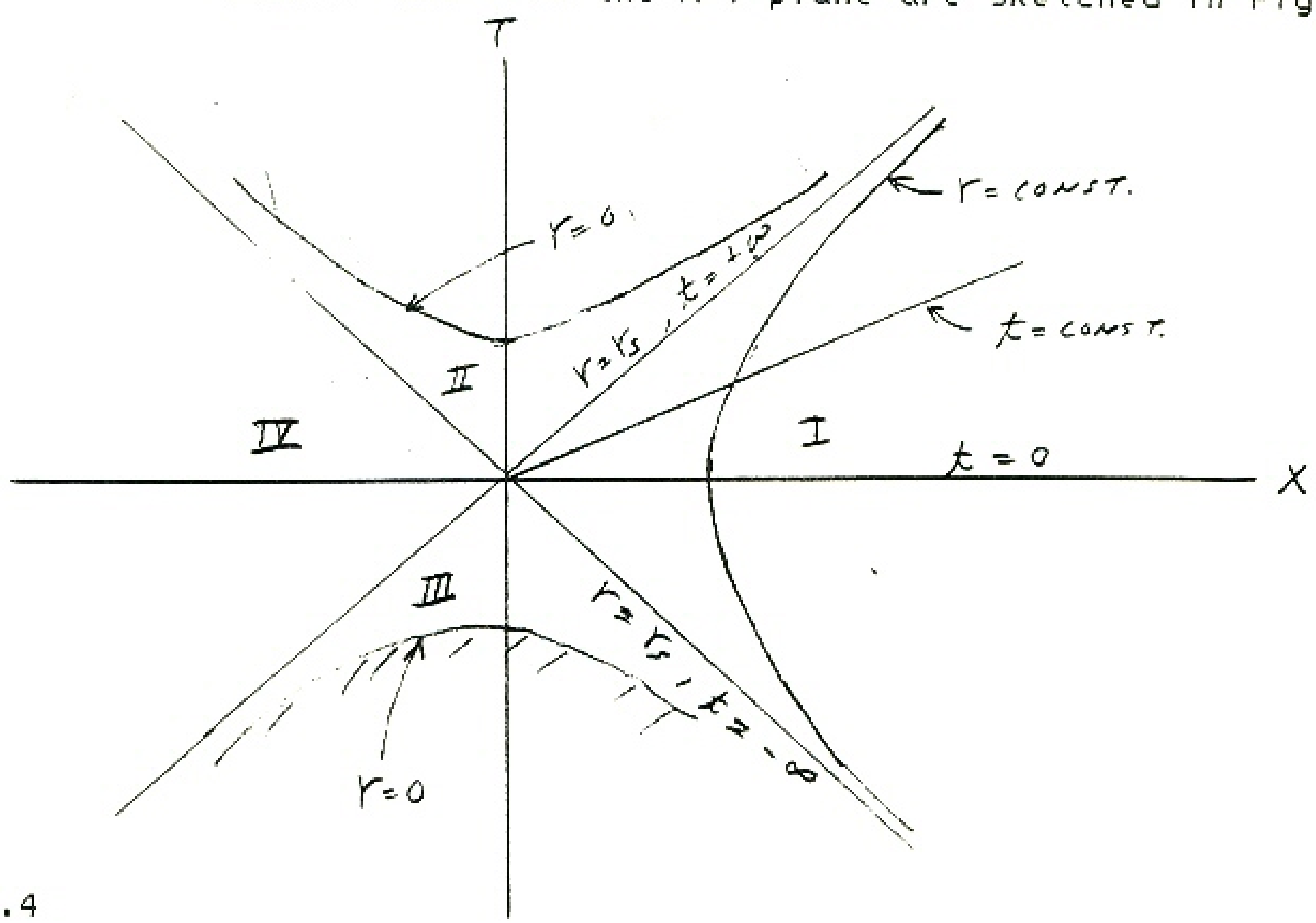


Fig.4

From Eq.(69a) we see that the lines $r$ = constant in the $r$-$t$ plane map onto hyperbolas $X^2 - T^2$ = constant in the X-T plane. For $r > r_s$ these hyperbolas have $X^2 - T^2$ = a positive constant and, for $r < r_s$ they have $X^2 - T^2$ = a negative constant. When $r = r_s$ the hyperbolas degenerate into the straight lines $X = \pm T$. The two hyperbolas corresponding to $r = 0$ are shown in the figure. Lines of constant $t$ in the $r$-$t$ plane correspond to lines in the X-T plane that pass through the origin and lie in the wedge shaped regions between $X = T$ and $X = -T$. No values of $t$ can be assigned to points of the X-T plane with $|T/X| > 1$. The lines $X = T$ and $X = -T$ divide the X-T plane into four regions labeled I, II, III and IV in the figure. It should be remembered that each point in the figure corresponds to a 2-sphere when $\theta$ and $\phi$ are allowed to vary over their ranges. The region of the r-t plane with $r > r_s$ is mapped into region I of the X-T plane. This is the part of the universe that we live in. Our coordinate transformation from (r,t) to (X,T) allows us to extend this region into the regions I, II, III, IV bounded by the curves r= 0 without crossing a singularity. The singularity at r = 0 remains in the metric of Eq.(68).

The world line of a photon has $d\tau = 0$. From Eq.(68) we see that for a radially moving photon $dT/dX = \pm 1$. These world lines are straight lines with a slope of 45 degrees. Outgoing photons have a positive slope and incoming photons have a negative slope. The worldlines of massive particles have $d\tau^2 > 0$ which implies $|dT/dX| > 1$ for the world lines of radially moving massive particles.

Now, consider the world line of a photon or massive particle moving radially inward from region I. It will pass through the line labeled ($r = r_s$, $t = +\infty$), enter region II and collide with the singularity at r = 0. On the other hand, the worldline of a radially outward moving photon or massive particle that starts in region II will collide with the singularity at r = 0 before it can pass into region I. Region II is called a <u>black hole</u>. Particles can fall into the black hole through the sphere at $r = r_s$, but no particle can ever emerge through this sphere into region I. It is as if the source of the gravitational field at r = 0 surrounded itself with a one-way spherical membrane of radius $r_s$ through which particles could pass inward but not outward.

The properties of region III are the "time reversed" properties of region I. No particle can enter region III from region I, but any world line in region III must have originated at the singularity at r = 0 and will ultimately pass through the line labeled ($r = r_s$, $t = -\infty$) into region I. Region III is called a <u>white hole</u>.

It is interesting to note that, although Einstein's equations are in variant under the time reversal transformation $t \to -t$, time irreversible behavior results in the solutions (that is, particles falling into but not out of black holes, and particles falling out of but not into white holes). Nature compensates for this by admitting both black and white hole solutions.

Region IV is an exact replica of region I. It represents another asymptotically flat region of spacetime. From the point of view of an observer in region I, it lies inside the radius $r = r_s$. Note that observers in regions I and IV would never be able to communicate. A photon directed from an observer in I toward IV would collide with the singularity at $r = 0$ before it could reach region IV and vice versa.

Let us consider a rocket ship falling into a black hole. The time measured by a clock carried in the ship is the proper time along its world line. The rocket ship would fall through the sphere at $r = r_s$ and strike the singularity at $r = 0$ in a finite proper time as Problem 5 below shows. Let us consider how this appears to a stationary observer at a distance $r \gg r_s$. For this observer the proper time is t. For this observer the rocket ship does not reach $r = r_s$ until $t = \infty$. Light signals sent from the ship to the distant observer would be shifted toward the red by both the Doppler effect and gravitational red shift. The distance observer would see the ship approaching but never reaching the sphere $r = r_s$ as the light from it was shifted toward the red.

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Problem 5.

Consider a particle released from rest at $r = r_0$. Show that it falls into the singularity at $r = 0$ in a proper time

$$\tau = (\pi r_0/2c)(r_0/r_s)^{1/2}$$

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## COSMOLOGICAL SOLUTIONS

When viewed on a small scale, the matter in the universe is distributed in a very nonuniform manner. It is concentrated in planets and stars which are concentrated in galaxies which in turn are concentrated in clusters of galaxies. However, from a large scale viewpoint it seems reasonable to disregard these inhomogeneities and to consider matter to be uniformly distributed throughout all of space. From what we know from astronomical observations there is no preferred direction in space, so it seems reasonable to assume the distribution of matter in the universe is both homogenous and isotropic. The question we wish to ask is this: What structure of spacetime is consistent with a homogenous isotropic distribution of matter?

First, we must choose a coordinate system. The most convenient choice is one in which each point moves with the matter located at that point. That is, each particle of matter is labeled with three spatial coordinates which it carries with it for all times. This is called a comoving coordinate system. The time coordinate is taken to be the proper time that would be measured by a clock carried by the particle. In the comoving system the velocity of matter is zero by definition, since the coordinates of a particle do not change with time. By our assumption, space is isotropic, so there is nothing to distinguish one spatial direction from another. It follows that the components $g_{0\nu}$ of the metric tensor must vanish. Therefore the metric can be written as

$$d\tau^2 = dt^2 - dl^2 \qquad (70a)$$

where

$$dl^2 = g_{ik}dx^i dx^k \qquad (70b)$$

is the spatial separation between neighboring points. In this section we employ natural units with $c = 1$.

Next, we must choose the structure of space. That is, we must choose $dl^2$ in such a way as to be consistent with our assumption that space is homogenous and isotropic. The simplist choice is

$$dl^2 = a^2(t)[dx^2 + dy^2 + dz^2]$$

$$= a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \qquad (71a)$$

where $a(t)$ is a scale factor that may depend on time. With this choice space is flat, although spacetime may be curved. Other choices of homogenous, isotropic spaces are the 3-sphere and the 3-pseudosphere with metrics given respectively by

$$dl^2 = a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] \qquad (71b)$$

$$dl^2 = a^2(t)[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] \qquad (71c)$$

These were discussed in Example (b) and Problem 4 of chapter 6. The 3-sphere has positive curvature, and the 3-pseudosphere has negative curvature.

We may use Eq.(71a,b,c) in Eq.(70) and write the space time metric as

$$d\tau^2 = dt^2 - a^2(t)[d\chi^2 + r^2(\chi,k)(d\theta^2 + \sin^2\theta\, d\phi^2)] \qquad (72a)$$

where $k = \pm 1$, 0 and

$$r(\chi,k) = (1/\sqrt{k})\sin\chi\sqrt{k} = \begin{cases} \sin\chi & , k = +1 \\ \chi & , k = 0 \\ \sinh\chi & , k = -1 \end{cases} \qquad (72b)$$

The curvature and Einstein tensors for these three metrics were worked out in Example (c) and Problems 4 and 5 of Chapter 6. The components of the Einstein tensors may be summarized as

$$G^0_0 = -3a'^2/a^2 - 3k/a^2$$

$$G^1_1 = G^2_2 = G^3_3 = -2a''/a - a'^2/a^2 - k/a^2 \tag{73}$$

where the prime denotes a derivative with respect to the argument t. All other components of the Einstein tensor vanish.

These are to be used in the Einstein field equations, Eq.(1), with the energy-momentum tensor given by

$$T^\mu_\nu = (\rho + U + P)u^\mu u_\nu - \delta^\mu_\nu P \tag{74}$$

Since the spatial components of the velocity 4-vector vanish in the comoving coordinates, $u^\mu = (1,0,0,0)$, and the nonvanishing components of the energy-momentum tensor are

$$T^0_0 = \rho + U = \epsilon \tag{75a}$$

$$T^1_1 = T^2_2 = T^3_3 = -P \tag{75b}$$

where $\epsilon$ is the total energy per unit volume including rest mass and thermal energy.

The two independent Einstein equations are

$$a'^2 + k = (8\pi G/3)\epsilon a^2 \tag{76a}$$

$$2a''/a + a'^2/a^2 + k/a^2 = -8\pi GP \tag{76b}$$

Eq.(76a) may be used in (76b) to rewrite it as

$$a'' = -4\pi G(P + \epsilon)a \quad = -\frac{4\pi G}{3}(\epsilon + 3P) \tag{76c}$$

Now consider

$$\frac{8\pi G}{3}\frac{d(\epsilon a^3)}{dt} = \frac{da(a'^2 + k)}{dt}$$

$$= a'(a'^2 + k) + 2aa'a''$$

$$= 8pGPa^2a'$$

$$= \frac{8\pi GP}{3} \frac{da^3}{dt} \qquad (77a)$$

We can write

$$\frac{d(\epsilon a^3)}{dt} + P\frac{da^3}{dt} = T\frac{dS}{dt} = T\frac{d(sa^3)}{dt} = 0 \qquad (77b)$$

We recognize this as the law of conservation of entropy with T the temperature, S the entropy and s the entropy per unit volume.

We now specialize to the case P = U = 0, so $\epsilon = r$. We call this the dust model since all of the energy is in the rest mass of the particles of the universe. Eq.(77b) gives

$$(8\pi G/3)\epsilon a^3 = \text{constant} = a_m \qquad (78)$$

It is convenient to change the independent variable from t to $\eta$ with the relation between the two given by

$$dt = a(t)d\eta \qquad (79)$$

With this change Eq.(76a) becomes

$$(da/d\eta)^2 + Ka2 = a_m a \qquad (80)$$

This is easily solved with the result

$$a(\eta) = \frac{a_m}{2}(1 - \cos\eta\sqrt{k}) \qquad (81a)$$

Eq.(79) can now be integrated to obtain

$$t = \frac{a_m}{2K}(\eta - \frac{\sin\eta\sqrt{K}}{\sqrt{K}}) \qquad (81b)$$

The constants of integration have been chosen so that a = 0 and t = 0 at $\eta$ = 0.

This completes the solution. With the scale factor determined, the metric is known. It is useful to make the change of variable of Eq.(79) in Eq.(72) and write the metric as

$$d\tau^2 = a^2(\eta)(d\eta^2 - dx^2 - r^2(x,K)(d\theta^2 + \sin^2\theta\, d\phi^2)) \qquad (82)$$

We shall now examine the three cases K = +1, -1 and 0 separately. For K = + 1,

$$a(\eta) = a_m/2\ (1 - \cos\eta) \qquad (83a)$$

$$t(\eta) = a_m/2\ (\eta - \sin\eta) \qquad (83b)$$

The scale factor starts at zero at the time t = 0, expands to a maximum value of $a_m$ (which is why we chose this notation) at a time $t = \pi a_m/2$ and then falls to zero at time $\pi a_m$. This universe is closed. The greatest spatial distance between any two points in the universe at the time t is $\pi a(t)$, the distance between a point at x = 0 and a point at x = $\pi$. The distance between any two points in the universe is zero at time t = 0 and again at $t = \pi a_m$. According to Eq.(78), The energy density (which is equal to the mass density) becomes infinite at these two times. The scalar curvature becomes infinite at these two times. The time t = 0 is called the time of the big bang, and the time $t = \pi a_m$ is called the time of the big crunch. Presumably the universe did not exist before the big bang and will no longer exist after the big crunch.

For $k = -1$

$$a(\eta) = a_m/2\ (\cosh\eta - 1) \qquad (84a)$$

$$t(\eta) = a_m/2\ (\sinh\eta - \eta) \tag{84b}$$

This universe is open. The scale factor is zero at time t and increases monotonically thereafter, ultimately increasing with the velocity of light. At $t = 0$ the distance between any two particles in the universe is zero and the mass density is infinite. There is a big bang but no big crunch.

For $k = 0$ we can take the limit of Eqs.(81a,b) as $k \to 0$ and obtain

$$a(\eta) = a_m/4\ \eta^2 \tag{85a}$$

$$t(\eta) = a_m/12\ \eta^3 \tag{85b}$$

We may eliminate $\eta$ between these two equations and obtain

$$a(t) = a_m{}^{1/3}(3t/2)^{2/3} \tag{85c}$$

This is an open universe with a big bang at $t = 0$ followed by monotonic expansion. Eqs.(85a,b,c) are the limiting forms of Eqs.(83a,b) and Eqs.(84a,b) for sufficiently small times.

Next, let us consider an observer at $x = 0$ observing a distant galaxy with coordinate x. The distance from the observer to the galaxy is $r = a(t)x$ according to Eq.(72). The velocity of recession of the galaxy is

$$v = dr/dt = a'x = a'/a\ r = Hr \tag{86a}$$

where

$$H = a'/a \tag{86b}$$

is called Hubble's constant. (It is not a constant of course. Its value at the present time is denoted by $H_0$.) The expansion of the universe can be detected by observation of the velocity of recession of distant galaxies by the Doppler shifts of spectral lines emitted by atoms in these galaxies. At the present time the best determination of $H_0$ from the measurement of $v$ and $r$ for a laerge number of galaxies is

$$H_0 = 50 \text{ km sec}^{-1} \text{ Mpc}^{-1} = 17 \times 10^{-19} \text{ sec}^{-1} \quad (87)$$

where we have used 1 megaparses = $3 \times 10^{19}$ km. We should emphasize that there is considerable uncertainty in this number because of the difficulty of determining the distance to galaxies. The number could be in error by as much as a factor of two.

The observed value of Hubble's constant can be used to estimate the age of the universe. Assuming that our universe is adequately described by a $k = 0$ solution, we use Eq.(85c) to find

$$H = a'/a = 2/3t \quad (88a)$$

$$\text{age of universe} = 3/2H_0 = 1.3 \times 10^{10} \text{ years} \quad (88b)$$

Whether our universe is open or closed, and which of the three models, $k = +1, -1, 0$, best describes our universe is a matter of considerable interest. In principle this question can be answered by observation. We divide Eq.(76a) through by $a^2$ and write

$$k/a^2 = 8\pi G\rho/3 - a'^2/a^2 = 8\pi G\rho/3 - H^2 \quad (89a)$$

$$k/a_0{}^2 H_0{}^2 = \rho/\rho_c - 1 \quad (89b)$$

where the critical mass density is

$$\rho_c = 3H_0{}^2/8\pi G = 5 \times 10^{-30} \text{ gm/cm}^3 \quad (89c)$$

If $\rho > \rho_c$, then $k = +1$ and the universe will eventually collapse. If $\rho < \rho_c$ or $\rho = \rho_c$, then $k = -1$ or $k = 0$ and the universe will expand forever. Unfortunately it is very difficult to determine the mass density of the universe. There is evidence that there exists matter in the universe that is not visible. The present indications are that $\rho$ differs from $\rho_c$ by no more than one or two orders of magnitude and may be either greater than or less than $\rho_c$. Some theoretical prejudices favor $\rho = \rho_c$.

Near the time of the big bang the matter in the universe is very highly compressed and the assumption that the energy is mostly rest mass energy is not valid. The energy density of radiation will be dominant. The energy per unit volume of radiation is $U = bT^4$, and the enropy per unit volume is $4bT^3/3$ where $b = \pi^2/15$. The conservaion of entropy, Eq.(77b), gives $sa^3$ = constant, which implies $aT$ = constant. The temperature varies inversely as $a(t)$. We may rewrite Eq.(76a) as

$$(da/d\eta)^2 + ka^2 = 8\pi G/3 \; \epsilon a^4 \qquad (90a)$$

Including both rest mass and radiant energy in $\epsilon$, we obtain

$$8\pi G/3 \; \epsilon a^4 = 8\pi G/3 \; (\rho + bT^4)a^4$$

$$= a_m a + C \qquad (90b)$$

where $a_m$ is given by Eq.~~(80)~~ 78 and

$$C = 8\pi G b(aT)^4/3 = \text{constant}. \qquad (90c)$$

Therefore

$$(da/d\eta)^2 + ka^2 = a_m a + C \qquad (90d)$$

For sufficiently early times, $a(t)$ is very small and $C \gg a_m a, ka^2$ and the solution is

$$a = C^{1/2}\eta \qquad (91a)$$

$$t = C^{1/2}\eta^2/2 \qquad (91b)$$

$$a = C^{1/4}(2t)^{1/2} \qquad (91c)$$

The expansion of the universe begins as the 1/2 power of t rather than the 2/3 power as given by Eq.(85c).

Before Hubble discovered the recession of distant galaxies in 1929 it was generally believed that the universe was stationary. Einstein was disappointed to find that the cosmological equations had no stationary solutions. In fact it is seen from Eq.(76c) that for any positive pressure and energy, the universe must be either expanding or contracting. Einstein looked for a way to modify the field equations to correct what he considered to be a deplorable situation. The modification he found was to add a term $\lambda g_{\mu\nu}$ to Eq.(1) so that it becomes

$$G_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \qquad (92)$$

in natural units. $\lambda$ is a constant called the <u>cosmological constant</u>. Since the covariant derivative of the metric tensor vanishes, the divergence of the left hand side vanishes after this addition as it did before. If $\lambda$ is sufficiently small, the deviation from Newton's theory in the small velocity, weak field limit would be negligible.

With the addition of this term, Eqs.(76a,c) are replaced by

$$a'^2 + k - \lambda a^2/3 = (8\pi G/3)\epsilon a^2 \qquad (93a)$$

$$a'' - \lambda a/3 = -(4\pi G/3)(\epsilon + 3P)a \qquad (93b)$$

These equations hava a stationary solution with $k = 1$ and $a' = a'' = 0$ given by

$$\lambda = 4\pi G(\epsilon + 3P), \quad 1 = 4\pi G(\epsilon + P)a^2 \qquad (94)$$

For a dust model with $P = 0$, the solution is a universe that is a 3-sphere of radius $a = 1/\surd\lambda$. These static solutions are unstable; a small perturbation of the solution grows exponentially with time. (See Problem 6.)

In 1922 the Russian mathematician Friedmann discovered the models of nonstatic universes discussed earlier in this section.

Einstein immediately recognized the importance of this discovery and concluded that there was no longer any need for the term $\lambda g_{\mu\nu}$ in the field equations. He is reported to have told George Gamov that the introduction of this term was the biggest blunder of his life. Hubble's discovery of the cosmological red shift a few years later gave further support to Friedmann's models. Never-the-less, there is no obvious reason why the term $\lambda g_{\mu\nu}$ should not be in the field equations. Why $\lambda$ should be so very small is a question that has been given no satisfactory answer at the present time.

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Problem 6.

Assuming a dust model with P = 0, Write Eq.(93b) as

$$a'' - \lambda a/3 = a_m/2a^2$$

Linearize this equation about the stationary solution by writing

$$a = a_0 + \delta a(t)$$

where

$$a_0 = 1/\sqrt{\lambda} = 3a_m/2$$

and retaining only the linear terms in $\delta a$. Show that there are solutions in which $\delta a(t)$ grows exponentially.

---

## SOME INHOMOGENOUS UNIVERSES

In this section we shall construct inhomogenous centrally symmetric solutions by patching together the Schwarzschild solution and the cosmological solutions obtained in earlier sections of this chapter. We begin by reexamining the notion of comoving coordinate systems. Consider the radial motion of a particle in a Schwarzschild metric. The Lagrangian is given by Eq.(28), and the constants of the motion are given by Eqs. (29a,b,c). For radial motion $p_\phi = 0$. The equations of motion are

$$dr/d\tau = (-\,Kr_s/r_m + r_s/r)^{1/2} \qquad (95a)$$

$$dt/d\tau = \frac{(1 - Kr_s/r_m)^{1/2}}{(1 - r_s/r)} \qquad (95b)$$

The solution of these equations and the definition of K and $r_m$ are given in Problem 3. When K = + 1, the particles are bound and $r_m$ is the maximum value of r.

Now, consider an infinite set of particles with all possible values of $r_m$ and K. Each particle is labeled with the value of $r_m$ and K that it carries. We shall make a coordinate transformation from $(t,r,\theta,\phi)$ to the comoving coordinates $(\tau,r_m,\theta,\phi)$. We can now write the Schwarzschild metric, Eq.(16), as

$$\begin{aligned} ds2 = {} & (1 - r_s/r)[(\partial t/\partial\tau)_{r_m} dt + (\partial t/\partial r_m)_\tau dr_m]^2 \\ & - (1 - r_s/r)^{-1}[(\partial r/\partial\tau)_{r_m} d\tau + (\partial r/\partial r_m)_\tau dr_m]^2 \\ & - r^2(r_m,\tau)d\Omega^2 \end{aligned} \qquad (96a)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{96b}$$

In order to avoid confusion we have replaced the differential of proper time $d\tau$ by ds on the left hand side of Eq.(96a) since $\tau$ is the time variable of the comoving coordinate system. The partial derivatives $(\partial r/\partial\tau)_{r_m}$ and $(\partial t/\partial\tau)_{r_m}$ are given by Eqs.(95a,b) since in those equations $r_m$ was treated as a constant and so the notation for ordinary derivatives was used. The partial derivatives $(\partial r/\partial r_m)_\tau$ and $(\partial t/\partial r_m)_\tau$ are to some extent arbitrary since the constants of integration $\tau_0$ and $t_0$ in the solutions of Problem 3 may be chosen to be arbitrary functions of $r_m$. We may use this arbitrariness to eliminate the coefficient of $dr_m d\tau$ in Eq.(96a) by choosing

$$(\partial t/\partial r_m)_\tau = \frac{1}{(1 - r_s/r_m)^2} \frac{(\partial r/\partial\tau)_{r_m}}{(\partial t/\partial\tau)_{r_m}} (\partial r/\partial r_m)_\tau \tag{97}$$

With this choice the metric becomes

$$ds^2 = d\tau^2 - \frac{r'^2(r_m,\tau)}{1 - kr_s/r_m} dr_m{}^2 - r^2(r_m,\tau)d\Omega^2 \tag{98a}$$

where

$$r'(r_m,\tau) = (\partial r/\partial r_m)_\tau \tag{98b}$$

These coordinates are called Novikov coordinates.

We shall arrive at this metric from a different starting point and at the same time obtain the field equations. In Problem 7 of Chapter 6 we found the Einstein tensor for the metric

$$ds^2 = d\tau^2 - e^\lambda dr_m{}^2 - e^\omega d\Omega^2 \tag{99}$$

where $\lambda = \lambda(r_m,\tau)$ and $\omega = \omega(r_m,\tau)$. The notation has been changed to agree with the notation we are currently using. Using the results of Problem 7, the Einstein field equations are

$$8\pi G T_0^0 = e^{-\omega} - e^{-\lambda}(\omega'' + 3\omega'^2/4 - \lambda'\omega'/2$$

$$+ \dot{\omega}^2/4 + \dot{\lambda}\dot{\omega}/2) \qquad (100a)$$

$$8\pi G T_1^1 = e^{-\omega} - e^{-\lambda}\omega'^2/4 + \ddot{\omega} + 3\dot{\omega}^2/4 \qquad (100b)$$

$$8\pi G T_2^2 = 8\pi G T_3^3 = - e^{-\lambda}[\omega''/2 + \omega'^2/4 - \lambda'\omega'/4]$$

$$+ \ddot{\lambda}/2 + \dot{\lambda}^2/4 + \ddot{\omega}/2 + \dot{\omega}^2/4 + \dot{\lambda}\dot{\omega}/4 \qquad (100c)$$

$$8\pi G T_0^1 = \omega'\dot{\omega}/2 - \dot{\lambda}\omega'/2 + \dot{\omega}' \qquad (100d)$$

For all other choices of $\mu$ and $\nu$, $G_\mu^\nu = 0$. Dots denote derivatives with respect to $\tau$ and primes denote derivatives with respect to $r_m$.

We shall assume a dust model, so that in a comoving coordinate system $T_0^0 = \rho$ and all other components vanish. Eq.(100d) can be written as

$$\dot{\lambda} = \dot{\omega} + 2\dot{\omega}'/\omega' = \partial/\partial\tau\ (\omega + \ln\omega'^2) \qquad (101a)$$

which may be integrated to obtain

$$e^{\lambda} = \frac{e^{\omega}\omega'^2}{4f^2(r_m)} \qquad (101b)$$

where $f(r_m)$ is an arbitrary function. When this is used in Eq.(100b), one obtains

$$(1 - f^2) + e^{\omega}(\omega + 3\dot{\omega}^2/4) = 0$$

$$= \frac{e^{-\omega/2}}{\dot{\omega}} \frac{\partial}{\partial\tau} (2(1 - f^2)e^{\omega/2} + e^{3\omega/2}\dot{\omega}^2/2) \qquad (102a)$$

This may be integrated to obtain

$$\dot{\omega}^2/2 + 2(1 - f^2) = F(r_m)e^{-3\omega/2} \tag{102b}$$

where $F(r_m)$ is an arbitrary function.

Using Eq.(101b) in (100c), one obtains

$$\frac{\partial}{\partial r_m}\{Eq.(102)\} = 0 \tag{103}$$

so nothing new is obtained from Eq.(100c).

Finally, using Eqs.(101b) and (102b) in (100a), we obtain

$$8\pi G\rho = \frac{e^{-3\omega/2}F'}{\omega'} \tag{104}$$

We now compare the metrics, Eq.(98a) and (99) and identify

$$e^{\omega} = r^2(r_m,\tau) \tag{105a}$$

Using this in Eq.(102b) gives

$$\dot{r}^2 = (f^2 - 1) + F/r \tag{105b}$$

Comparison with Eq.(95a) leads us to identify

$$f^2(r_m) = 1 - kr_s(r_m)/r_m \tag{106a}$$

$$F(r_m) = 2r_s(r_m) \tag{106b}$$

With these identifications, Eq.(101b) becomes

$$e^{\lambda} = \frac{r'^2(r_m,\tau)}{1 - kr_s/r_m} \tag{107}$$

This agrees with the coefficient of $dr_m^2$ in Eq.(98a). We have arrived at Eq.(98a) by two routes; first, by a transformation of coordinates from the Schwarzschild metric, and then by assuming Eq.(99) for the metric and integrating the field equations. The second derivation shows that it is not necessary to treat $r_s$ as a constant; it can be an arbitrary function of $r_m$. The relation between $r_s(r_m)$ and the mass density is given by Eq.(104) which may be written as

$$\frac{8\pi G\rho(r_m,\tau)}{3} = \frac{r_s'(r_m)}{(r^3(r_m,\tau))'} \tag{108}$$

We may choose $r_s(r_m)$ arbitrarily, determine $r(r_m,\tau)$ from the results of Problem 3 and then find $\rho(r_m,\tau)$ from Eq.(108). In this way we construct a solution of the field equations.

Two choices of $r_s(r_m)$ give particularly simple results. If we choose $r_s$ = constant, then $\rho = 0$ and the metric is the Schwarzschild metric given by either of the equivalent forms, Eq.(16) or (98). The other choice that gives a simple result is

$$r_s(r_m) = r_m^3/a_m^2 \tag{109a}$$

where $a_m$ is a constant. Then, from Problem 3

$$r(r_m,\tau) = r_m a(\tau)/a_m \tag{109b}$$

where

$$a(\tau) = a_m/2k\ (1 - \cos\eta\sqrt{k}) \tag{109c}$$

$$\tau - \tau_0 = a_m/2k\ (\eta - \sin\eta\sqrt{k}) \tag{109d}$$

Eq.(108) gives

$$\rho(r_m,\tau) = \rho(\tau) = \frac{3a_m}{8\pi G\, a^3(\tau)} \tag{110}$$

ON NEXT PAGE

$8\pi G a^3(\tau)$

Eq.(98) gives the metric

$$ds^2 = d\tau^2 - \frac{a^2(\tau)}{a_m^2}\left[\frac{dr_m^2}{1 - kr_m^2/a_m^2} + r_m^2 d\Omega^2\right] \qquad (111a)$$

A change of coordinates from $r_m$ to $x$ by

$$r_m = \frac{a_m \sin x\sqrt{k}}{\sqrt{k}} \qquad (111b)$$

puts the metric in the form

$$ds^2 = d\tau^2 - a^2(\tau)\left[dx^2 + \frac{\sin^2 x\sqrt{k}\; d\Omega^2}{k}\right] \qquad (111c)$$

which we recognize to be the same as Eq.(72), the metric for Freedmann's cosmological models.

A convenient way of constructing some models of inhomogenous universes is by joining together Schwarzschild solutions with $\rho = 0$ and Friedmann solutions with $\rho = \rho(\tau)$. The continuity conditions that must be imposed on $r(r_m,\tau)$ at the value of $r_m$ at which the solutions are joined may be obtained by inspection of the field equations. It is clear that $r(r_m,t)$ must be a continuous function of $r_m$, for otherwise $r'$ would have a delta function at the point of discontinuity and this would invalidate Eq.(108). On the other hand, at the value of $r_m$ at which the solutions are joined, $\rho$ and $r_s'$ have jump discontinuities, so according to Eq.(108) $r'$ may have a jump discontinuity also. Inspection of the field equations reveals that $r''$ does not appear, so there is no delta function that would invalidate these equations. Although $\omega'' = 2r''/r - 2r'^2/r^2$ does appear in Eq.(100c) it is cancelled by the $\omega''$ in $\lambda'\omega'/2 = \omega'' + \omega'^2/2 - \omega' f'/f$. It follows that the boundary condition at the value of $r_m$ where Schwarzschild and Friedmann solutions join is that $r(r_m, T)$ be continuous.

It is useful to make another change of coordinates from $(\tau, r_m, \theta, \phi)$ to $(\eta, x\ \theta, \phi)$. Using the results of Problem 3, we define

$$\tau(r_m, \eta) = h(r_m) g(\eta), \quad r(r_m, \eta) = r_m \dot{g}(\eta) \qquad (112a)$$

where

$$h(r_m) = \left(\frac{r_m^3}{r_s}\right)^{1/2} \qquad (112b)$$

$$g(\eta) = \frac{1}{2k}\left(\eta - \frac{\sin\eta\sqrt{k}}{\sqrt{k}}\right) \qquad (112c)$$

$$\dot{g}(\eta) = \frac{1}{2k}(1 - \cos\eta\sqrt{k}) \qquad (112d)$$

We define

$$dx = \frac{dr_m}{h(1 - kr_s/r_m)^{1/2}} \qquad (113)$$

In these coordinates the metric takes the form

$$ds^2 = h^2(x) g^2(\eta) \Big\{ \left[\frac{d\ln h}{dx}\, dx + \frac{d\ln g}{d\eta}\, d\eta\right]^2 - dx^2 \left[\frac{d\ln g}{d\eta} - r_m \frac{d\ln h}{dr_m} \frac{d\ln \dot{g}}{d\eta}\right]^2 - \frac{r_m^2 \dot{g}^2}{h^2 g^2}\, d\Omega^2 \Big\} \qquad (114)$$

When $r_s(r_m)$ is given by Eq.(109a), $h(r_m) = a_m$ = constant, Eq.(113) can be integrated to obtain

$$r_m = a_m \frac{\sin x\sqrt{k}}{\sqrt{k}} \tag{115}$$

and Eq.(114) reduces to Eq.(82) the, metric for the Friedmann cosmological models. When $r_s$ = constant, Eq.(113) can be integrated to obtain

$$r_m = \frac{kr_s}{\sin^2(x - x_0)\sqrt{k}/2} \tag{116}$$

In this case Eq.(114) for the metric has a rather complicated form, but we know that it is equivalent to the Schwarzschild metric.

We shall use these new coordinates to construct a model of a black hole embedded in a closed universe. We divide the range of $x$ into three regions-- a Friedmann region with $0 \leq x \leq x_1$, a Schwarzschild region with $x_1 \leq x \leq x_2$, and another Friedmann region with $x_2 \leq x \leq x_3$, where $x_3$ is the value of $x$ at which the universe closes. In these regions $r_s$ is given by

$$r_s(x) = \begin{cases} r_m^3(x)/a_m^2 & 0 \leq x \leq x_1 \\ r_{s1} = r_m^3(x_1)/a_m^2 = \text{const.} & x_1 \leq x \leq x_2 \\ r_m^3(x)/b_m^2 & x_2 \leq x \leq x_3 \end{cases} \qquad (117)$$

According to Eqs.(115) and (116), $r_m$ is given by

$$a_m \sin x \qquad 0 \leq x \leq x_1$$

$$r_{s1}$$

$$r_m(x) = \begin{cases} a_m \sin x & 0 \le x \le x_1 \\ \dfrac{r_{s1}}{\sin^2(x - x')/2} & x_1 \le x \le x_2 \\ b_m \sin(x - x'') & x_2 \le x \quad x_3 \end{cases} \qquad (118)$$

The constants $x'$, $x''$ and $b_m$ are to be chosen so that $r_s(x)$ and $r_m(x)$ are continuous at $x_1$ and $x_2$. When this is done, we find

$$r_m(x) = \begin{cases} a_m \sin x & 0 \le x \le x_1 \\ \dfrac{a_m \sin^3 x_1}{\sin^2(3x_1 - x)/2} & x_1 \le x \le x_2 \\ \dfrac{a_m \sin^3 x_1 \sin[x - 3(x_2 - x_1)/2]}{\sin^3(3x_1 - x_2)/2} & x_2 \le x \le x_3 \end{cases} \qquad (119)$$

In order to have $r_m = 0$ when $x = x_3$, indicating closure of the universe, we must have

$$x_3 = \pi + 3(x_2 - x_1)/2 \qquad (120)$$

In Fig.5a we have sketched the world lines of the particles of this universe in $(\eta, x)$ coordinates. They are vertical lines since each particle carries its $x$ coordinate with it. The world lines begin at $\eta = 0$, the time of the big bang, and terminate at $\eta = 2\pi$. We have used dotted lines in the Schwarzschild region to indicate that these are fictitious particles that carry coordinates but contribute nothing to the mass. In Fig.5b we have sketched $r_m(x)$. The coordinates $r$ and $\tau$ are given in terms of $\eta$ and $r_m(x)$ by the results of Problem 3. These have been used to sketch the world lines in $(\tau, r)$ coordinates in Fig.6

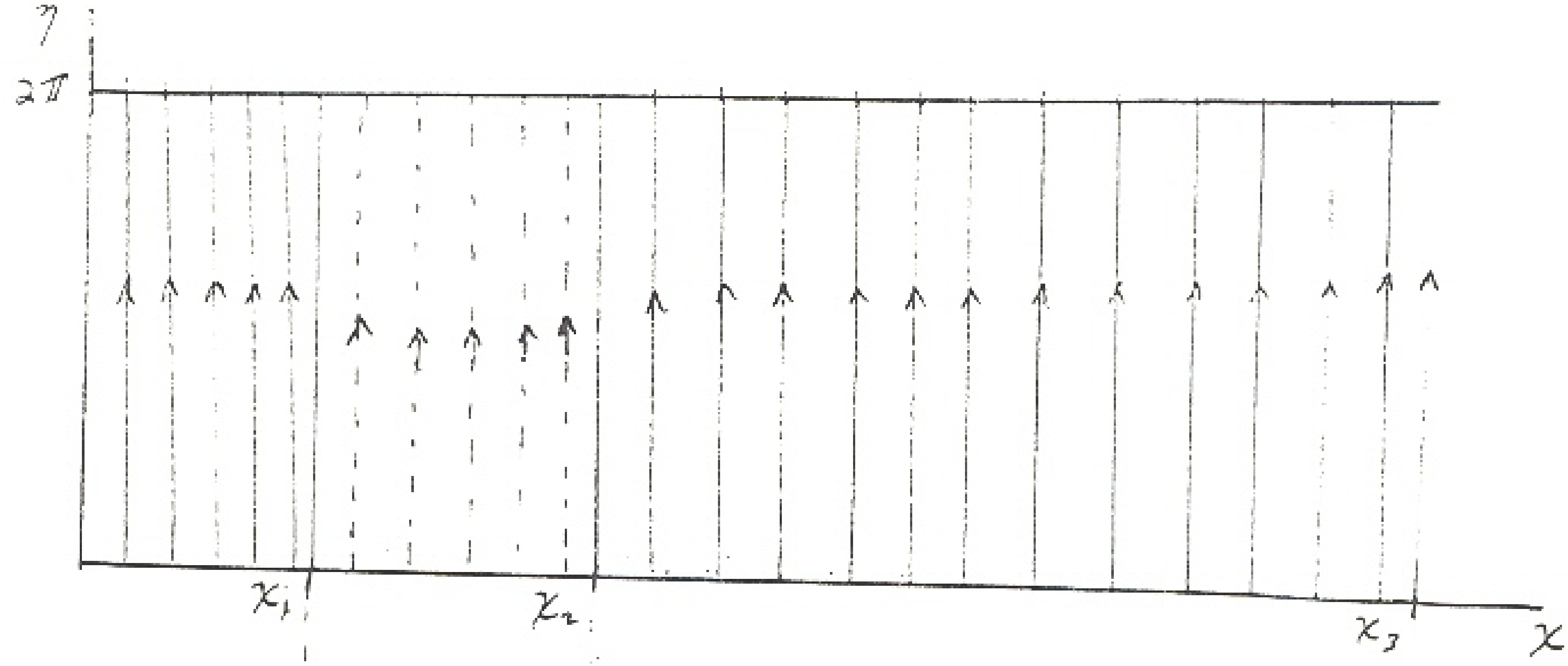


Fig.5a.

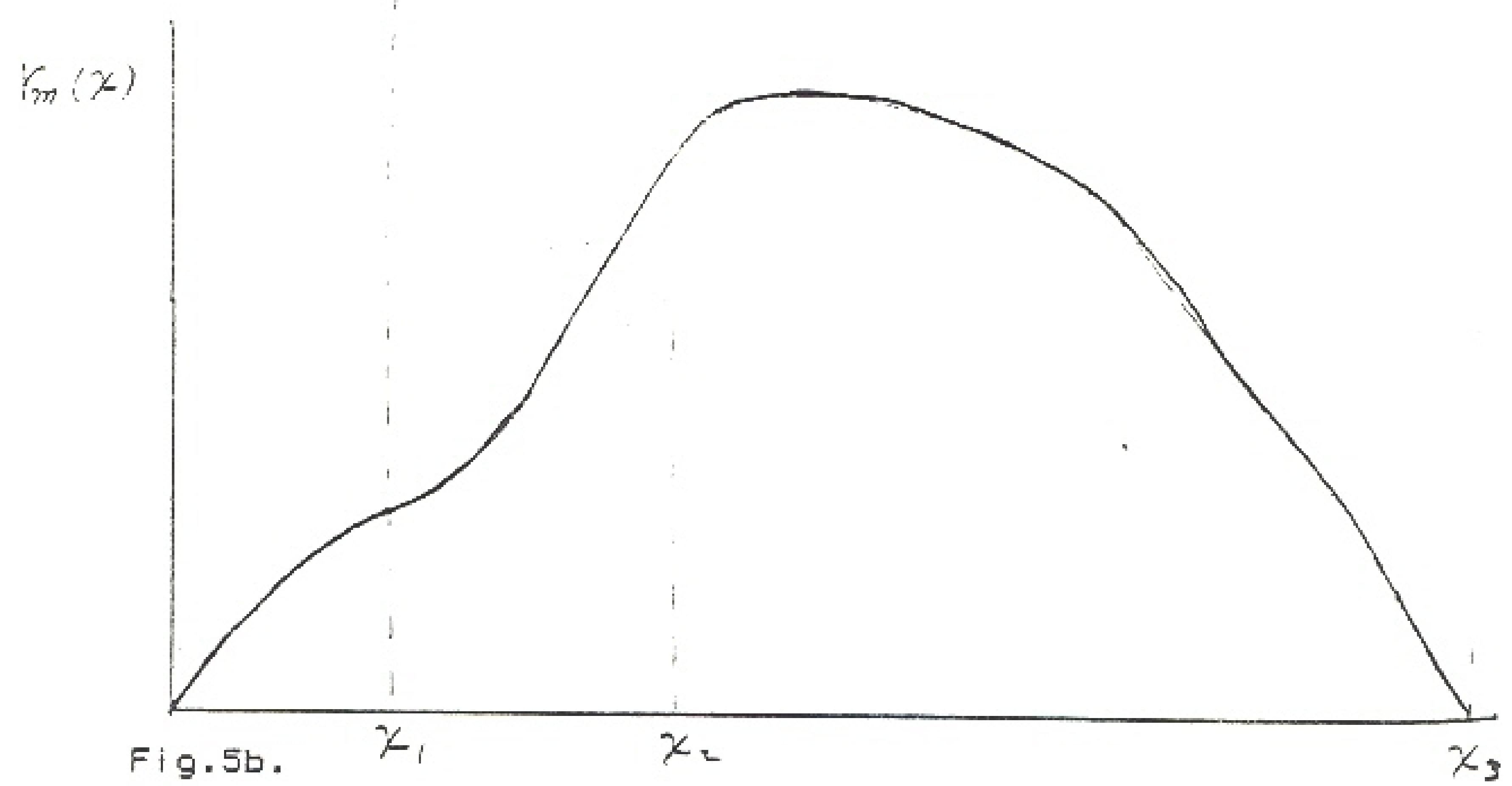


Fig.5b.

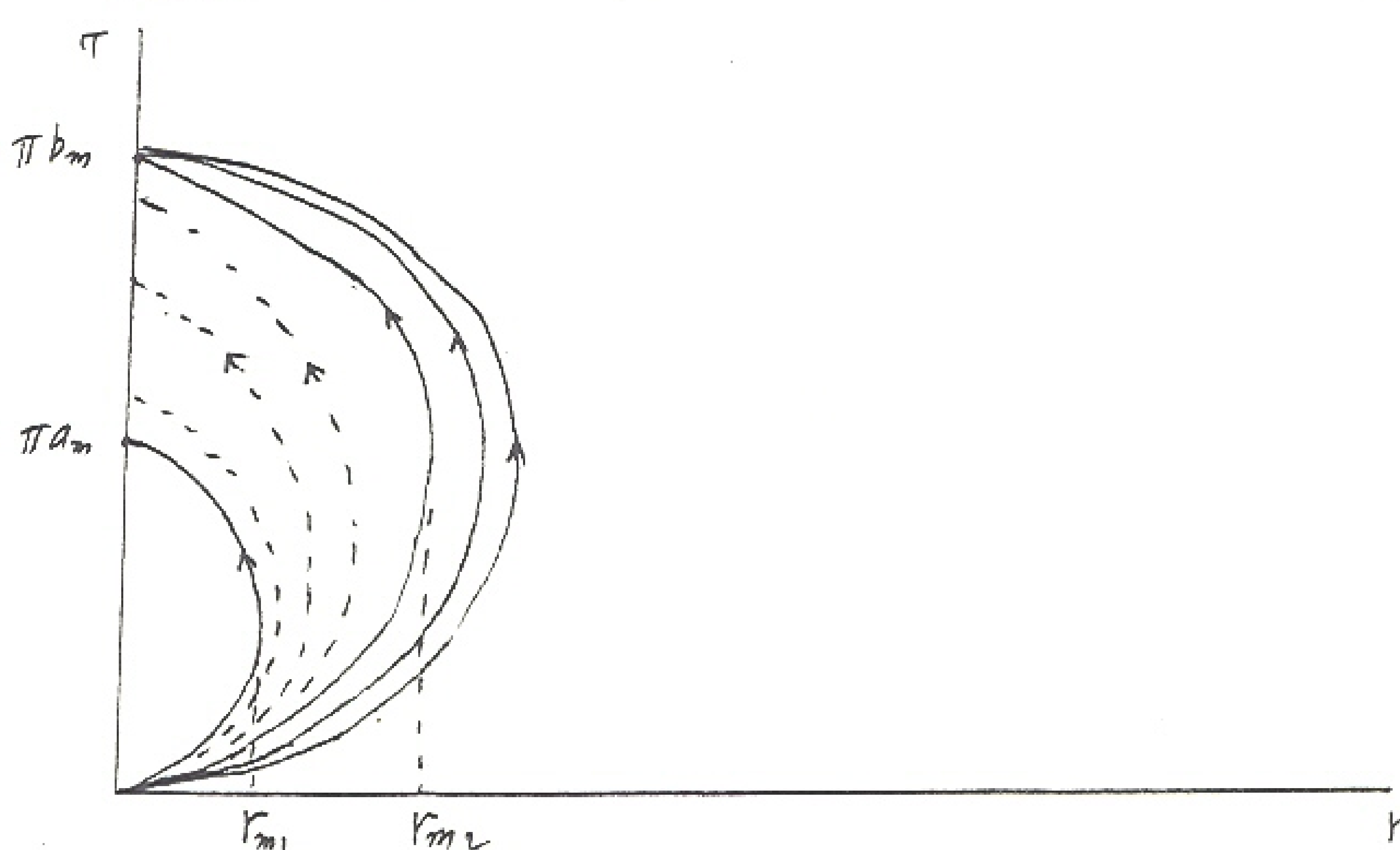

Fig.6.

We have taken $\tau_0 = 0$ in Problem 3 so all of the world lines in Fig.6 originate at $r = 0$, $\tau = 0$. The matter in the Friedmann region $0 \leq x \leq x_1$ collapses at the time $\tau = \pi a_m$. The remainder of the matter in the universe is in the second Friedmann region and collapses at the later time $\tau = \pi b_m$. To an observer in the second Friedmann region or the Schwarzschild region, the collapse of the matter in the first Friedmann region would appear as a collapse of a part of the universe to a black hole while the rest of the universe continues its expansion and collapse at the big crunch.

# SUMMARY OF GENERAL RELATIVITY

PARTICLE EQUATIONS OF MOTION GIVEN BY

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0 = \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$$

DOT DENOTES $\frac{d}{d\tau}$

$$L = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left(\frac{d\tau}{d\tau}\right)^2 = \frac{1}{2}$$

FIELD EQUATIONS ARE

$$\boxed{G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}}$$

WHERE

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} \quad ; \quad R = g^{\mu\nu} R_{\mu\nu}$$

$$R^\alpha{}_{\beta\mu\nu} = \frac{\partial \Gamma^\alpha_{\beta\mu}}{\partial x^\nu} - \frac{\partial \Gamma^\alpha_{\beta\nu}}{\partial x^\mu} + \Gamma^\eta_{\beta\mu}\Gamma^\alpha_{\nu\eta} - \Gamma^\eta_{\beta\nu}\Gamma^\alpha_{\mu\eta}$$

$$R_{\beta\nu} = \frac{\partial^2}{\partial x^\nu \partial x^\beta} \ln\sqrt{-g} - \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,\Gamma^\alpha_{\beta\nu}\right)$$

$$+ \Gamma^\eta_{\beta\alpha}\Gamma^\alpha_{\nu\eta}$$

FROM $\Gamma^\alpha_{\beta\alpha} = \frac{\partial}{\partial x^\beta} \ln\sqrt{-g} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\beta}\sqrt{-g}$

$$g = \text{DET } g_{\mu\nu}$$

SYMMETRIES AND IDENTITIES:

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu}$$

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = +R_{\mu\lambda\kappa\nu}$$

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu}$$

$$C_n = \frac{n^2(n^2-1)}{12}$$ = NUMBER OF INDEPENDENT COMPONENTS OF $R_{\lambda\mu\nu\kappa}$ IN AN $n$-DIMENSIONAL SPACE.

BIANCHI IDENTITIES:

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0$$

IT FOLLOWS THAT

$$R_{\mu\nu} = R_{\nu\mu}$$

$$G_{\mu\nu} = G_{\nu\mu}$$

$$G^{\mu\nu}{}_{;\nu} = 0$$

# SPECIAL SOLUTIONS



$$d\tau^2 = e^{\nu(r)} dt^2 - e^{+\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\Gamma^0_{01} = \nu'/2 \qquad \Gamma^2_{12} = \frac{1}{r}$$

$$\Gamma^1_{11} = \lambda'/2 \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^1_{00} = \nu'/2\, e^{\nu-\lambda} \qquad \Gamma^3_{13} = \frac{1}{r}$$

$$\Gamma^1_{33} = -e^{-\lambda} r \sin^2\theta \qquad \Gamma^3_{23} = \cot\theta$$

$$\Gamma^1_{22} = -e^{-\lambda} r$$

ALL OTHERS ARE ZERO

$$G^0_0 = e^{-\lambda}\left[\frac{1}{r^2} - \frac{\lambda'}{r}\right] - \frac{1}{r^2}$$

$$G^1_1 = e^{-\lambda}\left[\frac{1}{r^2} + \frac{\nu'}{r}\right] - \frac{1}{r^2}$$

$$G^2_2 = G^3_3 = e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'}{2r} + \frac{\nu'}{2r} - \frac{\lambda'\nu'}{4}\right]$$

IF $T^\mu_\nu = 0$ THEN

$$e^{\nu} = e^{-\lambda} = (1 - r_s/r)$$

$$d\tau^2 = (1 - r_s/r)\,dt^2 - \frac{dr^2}{(1-r_s/r)} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

= SCHWARZSCHILD METRIC

IF $T^\mu_\nu$ = ENERGY-MOMENTUM TENSOR FOR POINT CHARGE $q$

$$T^0_0 = T^1_1 = \frac{q^2}{8\pi r^2}, \quad \text{OTHERS} = 0$$

THEN

$$d\tau^2 = \left(1 - \frac{r_s}{r} + \frac{Gq^2}{c^4}\frac{1}{r^2}\right)dt^2 - \frac{dr^2}{\left(1 - r_s/r + \frac{Gq^2}{c^4}\frac{1}{r^2}\right)}$$

$$- r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$ = REISSNER-NORDSTROM METRIC

## ROBERTSON - WALKER METRIC

$$d\tau^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right\}$$

$$= dt^2 - a^2(t)\left\{d\chi^2 + r^2(\chi,k)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right\}$$

WHERE $r(\chi, k) = \frac{1}{\sqrt{k}} \sin\sqrt{k}\,\chi = \begin{cases} \sin\chi & k=+1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$

PRIME DENOTES DERIVATIVE WITH RESPECT TO ARGUMENT.

$$\Gamma^0_{11} = aa', \quad \Gamma^0_{22} = aa'r^2, \quad \Gamma^0_{33} = aa'r^2\sin^2\theta$$

$$\Gamma^1_{01} = a'/a, \quad \Gamma^1_{22} = -rr', \quad \Gamma^1_{33} = -rr'\sin^2\theta$$

$$\Gamma^2_{02} = a'/a, \quad \Gamma^2_{12} = r'/r, \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{03} = a'/a, \quad \Gamma^3_{13} = r'/r, \quad \Gamma^3_{23} = \cot\theta$$

OTHERS $= 0$

$$R^0_0 = 3a''/a, \quad R^1_1 = a''/a + 2a'^2/a^2 - 2r''/a^2 r$$

$$R^2_2 = R^3_3 = a''/a + 2a'^2/a^2 - r''/a^2 r - r'^2/a^2 r + \frac{1}{a^2 r^2}$$

$$G^0_0 = -3a'^2/a^2 - 3k/a^2$$

$$G^1_1 = G^2_2 = G^3_3 = -\frac{2a''}{a} - a'^2/a^2 - k/a^2$$